Equalities and Greater than or Equal to Constraints: Part I
Lecture V

I. Base Scenario

A. Example problem from text

$$\begin{aligned} &\min z = 4x_1 + x_2 \\ &s.t. \quad 3x_1 + x_2 = 3 \\ &\qquad 4x_1 + 3x_2 \geq 6 \\ &\qquad x_1 + 2x_2 \leq 4 \end{aligned}$$

or

$$\begin{aligned} &\min z - 4x_1 - x_2 + 0s_1 + 0s_2 \\ &s.t. \quad 3x_1 + x_2 = 3 \\ &\qquad 4x_1 + 3x_2 - s_1 = 6 \\ &\qquad x_1 + 2x_2 + s_2 = 4 \end{aligned}$$

What is the initial feasible solution to this problem?

1. Equation 1 has no slack or surplus (negative of the slack for greater than or equal to constraints) variable. Thus, there is no way to easily assign an initial solution.
2. Further, the easy solution for the second equation would be $s_1 = -6$ which violates the nonnegativity restrictions.

B. Artificial Variables

1. The techniques to remedy this problem involve adding a nonnegative variable to the left-hand side that has no obvious starting value.
2. The added variable will play the same role as a slack or surplus variable in providing an initial basic solution.
3. However, since the artificial variables have no meaning from the standpoint of the original problem (hence their name "artificial"), the procedure will be valid only when and if the variables are forced to zero.

C. The two methods used to force these artificial variables out of solution are the M-Technique (or penalty method) and the two-phase method. These methods are closely related.

II. M-Technique

A. The M-Technique involves adding a large penalty for violating the equality constraints ($+M$ for minimization and $-M$ for maximization).

B. Restated problem

$$\begin{aligned} \min z - 4x_1 - x_2 \quad & + MR_1 + MR_2 \\ s.t. \quad 3x_1 + x_2 \quad & + R_1 = 3 \\ 4x_1 + 3x_2 - s_1 \quad & + R_2 = 6 \\ x_1 + 2x_2 \quad & + s_2 = 4 \end{aligned}$$

Setting $x_1 = x_2 = s_1 = 0$ and letting the two artificial variables (R_1 and R_2) and the slack variable in the third constraint determine the initial starting solution ($R_1 = 3$, $R_2 = 6$, and $S_2 = 4$) we generate an initial solution. This solution is not "feasible" in the sense that all the constraints are satisfied. Specifically, the first and second constraints are by definition violated. Now, since the objective function coefficient for a basic variable is zero, we must solve the artificial variables out of the objective function. Noting

$$R_1 = 3 - 3x_1 - x_2 \text{ and } R_2 = 6 - 4x_1 - 3x_2 + s_1$$

Substituting these expressions into the objective function yields

$$\begin{aligned} z &= 4x_1 + x_2 + M(3 - 3x_1 - x_2) + M(6 - 4x_1 - 3x_2 + s_1) \\ &= 4x_1 + x_2 + 3M - 3Mx_1 - Mx_2 + 6M - 4Mx_1 - 3Mx_2 + Ms_1 \\ &= (4 - 7M)x_1 + (1 - 4M)x_2 + Ms_1 + 9M \\ \therefore\ z &= 9M \text{ if } R_1 = 3,\ R_2 = 6,\ s_2 = 4 \end{aligned}$$

The problem then becomes

$$\begin{aligned} \min z + (-4 + 7M)x_1 + (-1 + 4M)x_2 - Ms_1 \quad & = 9M \\ s.t. \quad 3x_1 + x_2 + R_1 \quad & = 3 \\ 4x_1 + 3x_2 - s_1 + R_2 \quad & = 6 \\ x_1 + 2x_2 + s_2 \quad & = 4 \end{aligned}$$

In tableau form

	x_1	x_2	s_1	R_1	R_2	s_2	RHS	
z	$(-4+7M)$	$(-1+4M)$	$-M$	0	0	0	$9M$	
R_1	3	1	0	1	0	0	3	$3/3 = 1$
R_2	4	3	-1	0	1	0	6	$6/4 = 3/2$
s_2	1	2	0	0	0	1	4	$4/1 = 4$

How do you choose the entry variable in the minimization problem? By the largest positive coefficient. What is *M*? Some arbitrarily large positive number. Thus, x_1 enters and R_1 leaves.