Canonical Forms of Linear Systems
Lecture IV

I. Systems of Equations with the Same Solution

A. Consistent equations and linear combinations

$$x_1 - x_2 + x_3 = 2 \ : \ E_1$$
$$2x_1 + x_2 - x_3 = 7 \ : \ E_2$$

1. The ordered set x_1=3, x_2=2, x_3=1 is said to be a solution of E_1 because the values substituted into E_1 yield 2=2. Thus, (3,2,1) is said to satisfy equation E_1. In general, for an arbitrary mxn (m equations and n unknowns) the solution (x_1^1,x_2^1, …x_n^1) is a solution of the i^{th} equation if
$$a_{i1}x_1^1 + a_{i2}x_2^1 + \cdots a_{in}x_n^1 = b_i$$
2. There are three possibilities for a given system.
   a. The system possesses a unique solution.
   b. The system possesses an infinite number of solutions.
   c. The system possesses no solution.
3. A system that possesses solutions whether unique or not is called consistent or solvable while a system containing no solution is inconsistent or unsolvable.
4. The aggregate of solutions of a system is called the solution set. If the system is inconsistent its solution set is said to be empty (or a null set).
   a. Given a system it is easy to construct new equations such that any solution of the original equations also solves the new equation.
$$\begin{array}{r} 2(\ \ x_1 - x_2 + x_3 = 2) \\ -3(2x_1 + x_2 - x_3 = 7) \\ \hline 2x_1 - 2x_2 + 2x_3 = 4 \\ -6x_1 - 3x_2 + 3x_3 = -21 \\ \hline -4x_1 - 5x_2 + 5x_2 = -17 \end{array}$$
   Notice that (3,2,1) solves the original system
$$3 - 2 + 1 = 2$$
$$2(3) + 2 - 1 = 7$$
   That point also solves the new system
$$-4(3) - 5(2) + 5(1) = 17$$
$$-12 - 10 + 5 = 17$$

5. A scheme for generating new equations is then

$$
\begin{array}{l}
k_1\left(a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1\right) \\
k_2\left(a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2\right) \\
\quad\vdots \\
\underline{k_m\left(a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m\right)} \\
d_1x_1 + d_2x_2 + \cdots d_nx_n = d
\end{array}
$$

$$
\begin{array}{l}
d_1 = k_1a_{11} + k_2a_{21} + \cdots k_ma_{m1} \\
d_2 = k_1a_{12} + k_2a_{22} + \cdots k_ma_{m2} \\
\quad\vdots \\
d_n = k_1a_{1n} + k_2a_{2n} + \cdots k_ma_{mn} \\
d = k_1b_1 + k_2b_2 + \cdots k_mb_m
\end{array}
$$

a. An equation formed in this manner is called a linear combination of the original equations. The numbers k_i are called multipliers or weights of the linear combination.

b. Writing these in detached form

$$
\begin{array}{ccc}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} & \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} & \begin{matrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{matrix} \\
\begin{bmatrix} d_1 & d_2 & \cdots & d_n \end{bmatrix} & d &
\end{array}
$$

6. Definition: If in a system of equations an equation is a linear combination of the other equations, it is said to be dependent upon them; the dependent equation is called redundant. A vacuous equation, i.e. an equation of the form

$$0x_1 + 0x_2 + \cdots 0x_n = 0$$

is also called redundant when it occurs in a single equation system. A system containing no redundancy is called independent

$$
\begin{array}{l}
5\left(x_1 + 3x_2 + 0x_3 = 4\right) \\
\underline{+\left(x_1 + x_2 - x_3 = 3\right)} \\
5x_1 + 15x_2 \qquad = 20 \\
\underline{x_1 + x_2 - x_3 = 3} \\
6x_1 + 16x_2 - x_3 = 23
\end{array}
$$

Therefore, in the system of equations

$$
\begin{array}{r}
x_1 + 3x_2 \qquad = 4 \\
x_1 + x_2 - x_3 = 3 \\
6x_1 + 16x_2 - x_3 = 23
\end{array}
$$

the last equation is redundant

B. How systems are solved

1. The usual "elimination" procedure for finding a solution of a system of equations is to augment the system by generating new equations by

taking linear combinations in such a way that certain coefficients are equal to zero. Taking a system of two equations as a starting point

$$x_1 - x_2 + x_3 = 2 \quad E_1$$
$$2x_1 + x_2 - x_3 = 7 \quad E_2$$

Let $E_3 = -2E_1 + E_2$

$$\begin{array}{rl} -2x_1 + 2x_2 - 2x_3 = -4 & E_1 \\ 2x_1 + x_2 - x_3 = 7 & E_2 \\ \hline 3x_2 - 3x_3 = 3 & E_3 \end{array}$$

Let $E_4 = \frac{1}{3} E_3$

$$x_2 - x_3 = 1 \quad E_4$$

Finally, let $E_5 = E_1 + E_4$

$$\begin{array}{rl} x_1 - x_2 + x_3 = 2 & E_1 \\ x_2 - x_3 = 1 & E_4 \\ \hline x_1 \qquad = 3 & E_5 \end{array}$$

Second example

$$\begin{array}{ll} A_1 & x_1 + 3x_2 = 4 \\ A_2 & x_1 + x_2 - x_3 = 3 \rightarrow A_2 - A_1 \\ A_3 & 6x_1 + 16x_2 - x_3 = 23 \rightarrow A_3 - 6A_1 \end{array}$$

These "row operations" on each equation yields

$$\begin{array}{ll} A_1^1 & x_1 + 3x_2 + 0x_3 = 4 \rightarrow A_1^1 + \frac{3}{2} A_2^1 \\ A_2^1 & 0x_1 - 2x_2 - x_3 = -1 \rightarrow -\frac{1}{2} A_2^1 \\ A_3^1 & 0x_1 - 2x_2 - x_3 = -1 \rightarrow A_3^1 - A_2^1 \end{array}$$

The second set of row operations then yields

$$\begin{array}{ll} A_1^2 & x_1 + 0x_2 - \frac{3}{2} x_3 = \frac{17}{2} \\ A_2^2 & 0x_1 + x_2 + \frac{1}{2} x_3 = -\frac{1}{2} \\ A_3^2 & 0x_1 + 0x_2 + 0x_3 = 0 \end{array}$$

2. Definition: A canonical system with an ordered subset of variables, called basic, is a system such that for each i, the i^{th} basic variable has a unit coefficient in the i^{th} equation and has zero coefficients elsewhere:

$$\begin{array}{l} x_1 \qquad + \bar{a}_{1,r+1} x_{r+1} + \cdots \bar{a}_{1,n} x_n = \bar{b}_1 \\ \quad x_2 \qquad + \bar{a}_{2,r+1} x_{r+1} + \cdots \bar{a}_{2,n} x_n = \bar{b}_2 \\ \qquad \vdots \\ \qquad x_r + \bar{a}_{r,r+1} x_{r+1} + \cdots \bar{a}_{r,n} x_n = \bar{b}_r \end{array}.$$

The preceding equation is in canonical form

$$\begin{array}{l} x_1 \qquad - \frac{3}{2} x_3 = \frac{17}{2} \\ \quad x_2 + \frac{1}{2} x_3 = -\frac{1}{2} \end{array}.$$

Deletion of an equation that is a linear combination of the other equations does not affect the solution set.

3. Definition: Two systems are called equivalent if one system may be derived from the other by inserting or deleting redundant equations or if one system may be derived from the other through a chain of systems linked to its predecessor by such an insertion or deletion.
4. Theorem: Equivalent systems have the same solution.

C. Elementary Operations

1. There are two simple, but important types of linear combinations that may be used to obtain equivalent systems.
   a. Replacing any equation E_t, by an equation kE_t with $k \neq 0$
   b. Replacing any equation E_t, by an equation $E_t + kE_i$ where E_i is any other equation of the system.

   {The "proof" of this proposition involves inserting the equation kE_t or $E_t + kE_i$ into any system. Notice that kE_t and $E_t + kE_i$ will always be redundant and, thus, could be eliminated}.

$$x_1 - x_2 + x_3 = 2 \quad E_1$$
$$2x_1 + x_2 - x_3 = 7 \quad E_2 \rightarrow E_2^1 = E_2 + E_1$$

$$x_1 - x_2 + x_3 = 2 \quad E_1^1 \rightarrow E_1^2 = E_1^1 - \tfrac{1}{3} E_2^1$$
$$3x_1 = 9 \quad E_2^1$$

$$-x_2 + x_3 = -1 \quad E_1^2 \rightarrow E_1^3 = -E_1^2$$
$$3x_1 = 9 \quad E_2^2 \rightarrow E_2^3 = \tfrac{1}{3} E_2^2$$

$$x_2 - x_3 = 1$$
$$x_1 = 3$$

2. Theorem 2: Corresponding to a sequence of elementary operations is an inverse sequence of elementary operations by which the given system can be obtained from the derived system

$$x_2 - x_3 = 1 \quad A_1 \rightarrow A_1^1 = -1A_1$$
$$x_1 = 3 \quad A_2 \rightarrow A_2^1 = 3A_2$$

$$-x_2 + x_3 = -1 \quad A_1^1 \rightarrow A_1^2 = A_1^1 + \tfrac{1}{3} A_2^1$$
$$3x_1 = 9 \quad A_2^1$$

$$x_1 - x_2 + x_3 = 2 \quad A_1^2$$
$$3x_1 = 9 \quad A_2^2 \rightarrow A_2^3 = A_2^2 - A_1^2$$

$$x_1 - x_2 + x_3 = 3$$
$$2x_1 + x_2 - x_3 = 7$$

3. Theorem 3: The rows of two equivalent systems in detached coefficient form can be obtained from the other by linear combinations.

$$x_1 - x_2 + x_3 = 2$$
$$2x_1 + x_2 - x_3 = 7$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 7 \end{array}\right] R_2 - 2R_1$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 3 \end{array}\right] R_2 \bullet 1/3$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & 1 \end{array}\right] R_1 + R_2$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

4. Theorem 4: If the i^{th} equation of a given system is replaced by a linear combination with multipliers k_i where $k_i \neq 0$, an equivalent system is obtained.

II. Canonical Systems

A. Solving Square Systems

1. The typical system from algebra or linear algebra involved the case where $m = n$. Such a system is said to be square. The solution of such a system typically involved solving for one variable in one equation and placing it into the next equation.

$$4x + 8y - 16z = 24$$
$$x - 3y - 4z = -9$$
$$2x - y - 3z = 8$$

$$4x + 8y - 16z = 24 \rightarrow x = 6 - 2y + 4z$$

$$(6 - 2y + 4z) - 3y - 4z = -9$$
$$-5y = -15$$
$$y = 3$$

$$2(6 - 2(3) + 4z) - (3) + 3z = 8$$
$$8z + 3 + 3z = 8$$
$$11z = 11$$
$$z = 1$$

$$x = 6 - 2(3) + 4 = 4$$

$$4 - 3(3) - 4(1) = -9$$

$$2(4) - (3) + 3(1) = 8$$

2. This procedure is nothing different than a sequence of elementary operations

$$\left[\begin{array}{ccc|c} 4 & 8 & -16 & 24 \\ 1 & -3 & -4 & -9 \\ 2 & -1 & 3 & 8 \end{array}\right] R_1 \bullet 1/4$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 6 \\ 1 & -3 & -4 & -9 \\ 2 & -1 & 3 & 8 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 04 & 6 \\ 0 & -5 & 0 & -15 \\ 0 & -5 & 11 & 4 \end{array}\right] \begin{array}{l} \\ R_2 \bullet 1/5 \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 11 & 11 \end{array}\right] R_3 \bullet 1/11$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \begin{array}{r} x_1 = 6 - 2x_2 + 4x_3 = 4 \\ x_2 = 3 \\ x_3 = 1 \end{array}$$

B. Pivotal Reduction of a General System

1. Instead of a square system, more generally, we have a system of m equations and n unknowns where $m \leq n$.

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m \end{array}$$

2. We are then interested in replacing this system with a canonical system:

$$x_2 + x_3 + x_4 = 5$$
$$x_1 - 2x_2 - x_3 + x_4 = 2$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 5 \\ 1 & -2 & -1 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 \to R_2 \\ R_2 \to R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 5 \end{array}\right] \begin{array}{l} R_1 + 2R_2 \\ \end{array}$$

in this case the "pivot element" we want to use is $1x_2$ and eliminate x_2 from the other equations using elementary row operations. In this solution x_1 and x_2 are the basic variables-the variables with a non-zero solution. x_3 and x_4 are non-basic variables taken to be zero.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 12 \\ 0 & 1 & 1 & 1 & 5 \end{array}\right]$$

3. Definition: A pivot operation consists of m elementary operations that replace a system by an equivalent system in which a specified variable has a coefficient of unity in one equation and zero elsewhere.
   a. Select a term $a_{rs}x_s$ in the system such tat $a_{rs} \neq 0$, called the pivot term.
   b. Replace the $r^{\underline{th}}$ equation by the $r^{\underline{th}}$ equation multiplied by $1/a_{4s}$
   c. For each $i = 1,2,\ldots m$ except $i = r$, replace the $i^{\underline{th}}$ equation by the sum of the $i^{\underline{th}}$ equation and the replaced $r^{\underline{th}}$ equation multiplied by $-a_{ir}$.
4. We refer to the resulting solution as the "reduced system".
5. Since the reduced system is formed from the original system using m elementary operations, it follows that the reduced system is:
   a. Formed from linear combinations of the original system, and
   b. Equivalent to the original system.

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m \end{array}$$

implies

$$
\begin{aligned}
x_1 \qquad\qquad &+ \bar{a}_{1,r+1}x_{r+1} + \bar{a}_{1,r+2}x_{r+2} + \cdots \bar{a}_{1,n}x_n = \bar{b}_1 \\
x_2 \qquad &+ \bar{a}_{2,r+1}x_{r+1} + \bar{a}_{2,r+2}x_{r+2} + \cdots \bar{a}_{2,n}x_n = \bar{b}_2 \\
&\vdots \\
x_r &+ \bar{a}_{r,r+1}x_{r+1} + \bar{a}_{r,r+2}x_{r+2} + \cdots \bar{a}_{r,n}x_n = \bar{b}_r \\
&0\,x_{r+1} + \quad 0\,x_{r+2} + \cdots\ 0\,x_n \ = \bar{b}_{r+1} \\
&\vdots \\
&0\,x_{r+1} + \quad 0\,x_{r+2} + \cdots\ 0\,x_n \ = \bar{b}_m
\end{aligned}
$$

Note that if $\bar{b}_{r+1}$ through $\bar{b}_m$ are not zero, the system is inconsistent.

C. Canonical Systems

$$
\begin{aligned}
x_1 \qquad\qquad &+ \bar{a}_{1,r+1}x_{r+1} + \bar{a}_{1,r+2}x_{r+2} + \cdots \bar{a}_{1,n}x_n = \bar{b}_1 \\
x_2 \qquad &+ \bar{a}_{2,r+1}x_{r+1} + \bar{a}_{2,r+2}x_{r+2} + \cdots \bar{a}_{2,n}x_n = \bar{b}_2 \\
&\vdots \\
\underbrace{\qquad\qquad x_r}_{\text{Dependent or Basic variables}} &+ \underbrace{\bar{a}_{r,r+1}x_{r+1} + \bar{a}_{r,r+2}x_{r+2} + \cdots \bar{a}_{r,n}x_n}_{\text{Independent or Nonbasic variables}} = \bar{b}_r
\end{aligned}
$$

D. Uniqueness of a Canonical System
   1. The fundamental property of a canonical system resulting from the reduction process is that for any other system with the same solution set a reduction can be affected using the same pivotal variables and the resulting canonical system will be identical if the equations and variables are reordered.
   2. Theorem 1: There is at most one equivalent canonical system with a fixed set of basic variables.
   3. Lemma 1: Any equation can either be generated by a unique linear combination of the equations of a canonical system (the weights being the coefficients on the basic variables in the equation) or no linear combination exists.

E. The special solution obtained by setting the independent variables equal to zero and solving for the dependent variables is called the basic solution.

F. A basic solution is degenerate if the values of one or more of the dependent (basic) variables equals zero.

G. In accordance with the special usage in linear programming, the term basis refers to an ordered set of columns of the original independent system (in detached coefficient form) corresponding to the ordered set of basic variables of a canonical equivalent. The columns of the basis are called basic columns:

$$
\begin{aligned}
x_2 + x_3 + x_4 &= 5 \\
x_1 - 2x_2 - x_3 + x_4 &= 2
\end{aligned}
$$

$$
\text{basis} \rightarrow \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}
$$