AEB 6533
Some Examples of Nullspaces

I. Starting with a simple 2*3 example, assume that we have the matrix equation

$$Ax = b \Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

A. Note that by row operations the A matrix can be transformed to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

B. This expression implies the following homogeneous relationships:

$$x_1 + 2x_3 = 0$$

$$x_2 + x_3 = 0$$

1. Setting $x_3 = 1$ yields

$$x_1 = -2$$

$$x_2 = -1$$

2. Or in vector form:

$$z = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

C. Next we confirm that z is the nullspace. We do this by confirming that both vectors of the A matrix are orthogonal to the nullspace:

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = -2 + 1 + 1 = 0$$

$$\begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0$$

D. Implications of the nullspace:

1. Start by defining a feasible solution:

$$Ax = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 1 & 6 \end{array} \right) = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 11 \\ 0 & 1 & 1 & 6 \end{array} \right)$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 11 - 6 + 0 \\ 6 + 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

2. Next, we would like to generate another feasible point based on this solution and the nullspace matrix:

$$x = x^* + Zp \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} (p)$$

Letting $p = 5$ yields

$$x = x^* + Zp \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} (5) = \begin{pmatrix} 11-10 \\ 6-5 \\ 0+5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

3. Checking the original solution:

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1-1+5 \\ 0+1+5 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

II. A slightly more complicated example, the case of a 2*4 constraint matrix.

$$Ax = b \Rightarrow \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \left(\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 4 \\ 0 & 1 & -1 & -1 & 3 \end{array} \right) = \left(\begin{array}{cccc|c} 1 & 0 & 1 & 7 & -5 \\ 0 & 1 & -1 & -1 & 3 \end{array} \right)$$

A. This solution gives us the complete nullspace. First, if we substitute the homogeneous solution for the particular solution above, we have

$$Ax = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 7 \\ 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

$$x_1 = -x_3 - 7x_4$$
$$x_2 = x_3 + x_4$$

B. This system implies several different solutions by setting x_3 and x_4 to different values. For example, if we set

$$x_3 = x_4 = 1 \Rightarrow \begin{matrix} x_1 = -1-7 = -8 \\ x_2 = 1+1 = 2 \end{matrix}$$

Note that this is a valid vector in nullspace:

$$\begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -8 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -8+6-2+4 \\ 0+2-1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

However, in this case we need to determine two nullspace vectors. In addition, we want the two vectors to be linearly independent. Thus, we choose two solutions of the homogeneous set of equations:

$$x_3 = 1, x_4 = 0 \Rightarrow \begin{matrix} x_1 = -1+0 = -1 \\ x_2 = 1+0 = 1 \end{matrix} \Rightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$x_3 = 0, x_4 = 1 \Rightarrow \begin{matrix} x_1 = 0-7 = -7 \\ x_2 = 0+1 = 1 \end{matrix} \Rightarrow \begin{pmatrix} -7 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

C. Checking this nullspace:

$$\begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -7 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1+3-2+0 & -7+3+0+4 \\ 0+1-1+0 & 0+1+0-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

D. Starting from our original solution, we can derive another feasible point as:

$$\begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -5 \\ 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5+9+0+0 \\ 0+3+0+0 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

We construct the variation:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & -7 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Letting

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & -7 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1-7 \\ 1+1 \\ 1+0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} -13 \\ 5 \\ 1 \\ 1 \end{pmatrix}$$

Finally, we demonstrate that the new point is still feasible:

$$\begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -13 \\ 5 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -13+15-2+4 \\ 0+5-1-1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$