Matrix Factorizations

The LU Factorization

The matrix factorization $A = LU$ involves reducing a matrix into a lower diagnal matrix L and an upper diagonal matrix U. The LU factorization is computed by a set of elementry row operations. Taking the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{bmatrix}$$

the first elementry matrix is

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

The resulting matrix is

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 6 \\ 0 & -1 & 9 \end{bmatrix}.$$

The second factorization is then

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

yielding the upper triangular matrix

$$A_2 = U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}.$$

The lower triangular matrix is then derived as

$$L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}.$$

The usefulness of the decomposition is twofold. First, the LU factorization is a quick way to solve a large system of equations. First, note that the solution of the system of linear equations $Ax = b$ for $b = (1, 5, 3)'$ is $x = (-5/2, 4, 1)'$. If we start with the augmented system of equations

$$[A|b] = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 6 & 2 & 6 & 5 \\ -4 & -3 & 9 & 3 \end{array}\right]$$

operation by the same sequence of elementry row operations yields

$$[A_1|b_1] = \begin{bmatrix} 2 & 1 & 0 & | & 1 \\ 0 & -1 & 6 & | & 2 \\ 0 & -1 & 9 & | & 5 \end{bmatrix}.$$

the second elementry row operation is then

$$[A_2|b_2] = \begin{bmatrix} 2 & 1 & 0 & | & 1 \\ 0 & -1 & 6 & | & 2 \\ 0 & 0 & 3 & | & 3 \end{bmatrix}.$$

In this system is the solution of the original equations taking the last equation first

$$3x_3 = 3$$
$$x_3 = 1.$$

Given that $x_3 = 1$ the nest equation up becomes

$$-x_2 + 6(1) = 2$$
$$x_2 = 4.$$

The first equation then completes the solution

$$2x_1 + (4) = 1$$
$$x_1 = -\frac{5}{2}.$$

The second major use is in the cholesky decomposition. If a matrix is symmetric and positive definite it can be decomposed so that $A = PP'$. Using the matrix

$$A = \begin{bmatrix} 7 & 3 & 1 \\ 3 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

First, we need to determine whether the matrix is positive definite. The eigenvalues of the matrix are 2.5572, 1.4841, and 8.9587 so the matrix is positive definite. Following the procedure above we derive the L matrix as

$$L = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.4286 & 1.0000 & 0.0000 \\ 0.1429 & -0.1579 & 1.000 \end{bmatrix}$$

with the upper triangular matrix as

$$U = \begin{bmatrix} 7.0000 & 3.0000 & 1.0000 \\ 0.0000 & 2.7143 & -0.4286 \\ 0.0000 & 0.0000 & 1.7895 \end{bmatrix}.$$

Noting that the diagonal of the L matrix is 1 and the diagonal matrix of the U matrix are greater than zero, suppose that we constructed the matrix

$$D^{-1} = \begin{bmatrix} 0.1429 & 0.0000 & 0.0000 \\ 0.0000 & 0.3684 & 0.0000 \\ 0.0000 & 0.0000 & 0.5588 \end{bmatrix}.$$

which is one over the diagonal elements of the U matrix. Multiplying $D^{-1}U$ yields L'.