Lecture XII
Linear Equality and Inequality Constraints

I. Linear Equality Constraints

A. The general optimization problem for the linear equality constraints can be stated as:

$$LEP \ \min_x f(x)$$
$$st \quad Ax = b$$

B. Like the unconstrained problem, we are again looking for the point where the projected gradient vanishes. However, this time instead of searching over dimension n, we only have to search over dimension n-t where t is the number of nonredundant equations in A.

1. In the vernacular of the problem, we want to decompose the vector x into a range-specific portion which is required to solve the constraints and a null-space portion which can be varied.

   a. Specifically,

   $$x = Y x_Y + Z x_Z$$

   where $Y x_Y$ denotes the range-specific portion of x and $Z x_Z$ denotes the null-space portion of x.

   b. By the definition of the null-space, we know that

   $$A Y x_Y^* = b$$

   if x^* is feasible, since AZ=0.

2. The numerical example given in the text uses a single constraint

   $$x_1 + x_2 + x_3 = 3$$

   There are several ways to derive the null-space of a matrix. The null-space matrix is

   $$Z = \begin{pmatrix} -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ \frac{3+\sqrt{3}}{6} & -\frac{3-\sqrt{3}}{6} \\ -\frac{3-\sqrt{3}}{6} & \frac{3+\sqrt{3}}{6} \end{pmatrix}$$

   A numerical equivalent to this matrix can be computed using Gauss. Specifically,

```
(gauss) a={1 1 1};
(gauss) null(a);

   -0.57735027    -0.57735027
    0.78867513    -0.21132487
   -0.21132487     0.78867513
```

Thus, the solution of the constraint can be expressed as:

$$x^* = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ \frac{3+\sqrt{3}}{6} & -\frac{3-\sqrt{3}}{6} \\ -\frac{3-\sqrt{3}}{6} & \frac{3+\sqrt{3}}{6} \end{pmatrix} x_Z^*$$

C. The step for the model algorithm must be selected so that

$$Ap_k = 0$$

Therefore, p_k must be a linear combination of the columns in Z:

$$Ap_k = 0 \Leftrightarrow p_k = Z\, p_z$$

In this case, a n*1 vector p_z is "mapped" into p_k by the null-space matrix, Z.

D. **Algorithm LE** (Model algorithm for solving LEP)

LE1. [Test for Convergence] If the conditions for convergence are satisfied, the algorithm terminates with x_k.

LE2. [Compute a feasible search direction] Compute a nonzero vector p_z, the unrestricted direction of the search. The actual direction of the search is then

$$p_k = Z\, p_z\,.$$

LE3. [Compute a step length] Compute a positive α_k, for which $f(x_k + \alpha_k p_k) < f(x_k)$.

LE4. [Update the estimate of the minimum] $x_{k+1} = x_k + \alpha_k p_k$ and go back to LE1.

E. Computation of the Search Direction

1. As is often the case in this course, the question of the search direction starts with the second order Taylor series expansion. As in the unconstrained case, we derive the approximation of the objective function around some point x_k as

$$f(x_k + p_k) = f(x_k) + \nabla_x f(x_k)\, p_k + \frac{1}{2} p_k{}' \nabla^2_{xx} f(x_k)\, p_k + \cdots$$

Substituting only feasible steps for all possible steps, we derive the same expression in terms of the null-space:

$$f(x_k + Zp_Z) = f(x_k) + \nabla_x f(x_k)\, Zp_Z + \frac{1}{2} p_Z{}' Z' \nabla^2_{xx} f(x_k)\, Zp_Z + \cdots$$

a. Solving for the projection based on the Newton-Raphson concept, we derive much the same steps as the constrained optimization problem:

i. Steepest Descent

$$p_k = -ZZ' \nabla_x f(x_k)$$

ii. Newton-Raphson

$$p_k = -\left(Z' \nabla^2_{xx} f(x_k) Z\right)^{-1} Z' \nabla_x f(x_k)$$

II. Linear Inequality Constraints

A. The general optimization problem with linear inequality constraints can be written as:

$$LIP \quad \min_{x} f(x)$$
$$st \quad Ax \geq b$$

B. This problem differs from the linearly constrained problem by the fact that some of the constraints may not be active at a given iteration, or may become active at the next iteration.

C. **Algorithm LI**

LI1. [Test for convergence] If the conditions for convergence are met at x_k, terminate.

LI2. [Choose which logic to perform] Decide whether to continue minimizing in the current subspace or whether to delete a constraint from the working set. If a constraint is to be deleted go to step LI6. If the same working set is to be retained, go on to step LI3.

LI3. [Compute a feasible search direction] Compute a vector p_k by applying the null-space equality

$$p_k = Z_k p_Z$$

LI4. [Compute a step length] Compute α, in this case, we must dtermine if the optimum step length will violate a constraint. Specifically α is equal to the traditional α_k or $\min(\gamma_I)$ which is defined as the minimum distance to a constraint. If the optimum step is less than the minimum distance to another constraint, then go to LI7, otherwise go to LI5.

LI5. [Add a constraint to the working set] If the optimum step is greater than the minimum distance to another constraint, then you have to add, or make active, the constraint associated with γ_i. After adding this constraint, go to L17.

LI6. [Delete a constraint] If the marginal value of one of the Lagrange multipliers is negative, then the associated constraint is binding the objective function suboptimally and the constraint should be eliminated. Delete the constraint from the active set and return to LI1.

LI7. [Update the estimate of the solution]. $x_{k+1} = x_k + \alpha_k p_k$ and go back to LI1.

D. A significant portion of the discussion in the LIP algorithm centered around the addition or elimination of an active constraint.

1. The concept is identical to the minimum ratio rule in linear programming. Specifically, the minimum ratio rule in linear programming identifies the equation (row) which must leave solution in order to maintain feasibility. The rule is to select that row with the minimum positive ratio of the current right hand side to the a_{ij} coefficient in the matrix.

2. In the nonlinear problem, we define

$$\gamma_i = \left\{ \frac{b_i - a_i' x}{a_i' p} \; a_i p < 0 \right\}$$