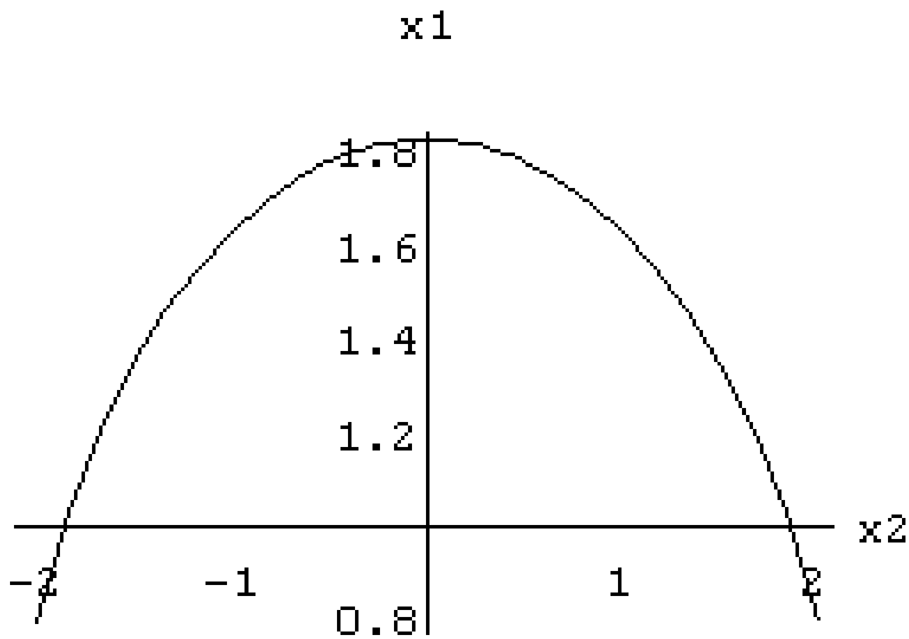
Lecture IX
Necessary and Sufficient Conditions under Nonlinear Constraints

I. A Simple Constraint

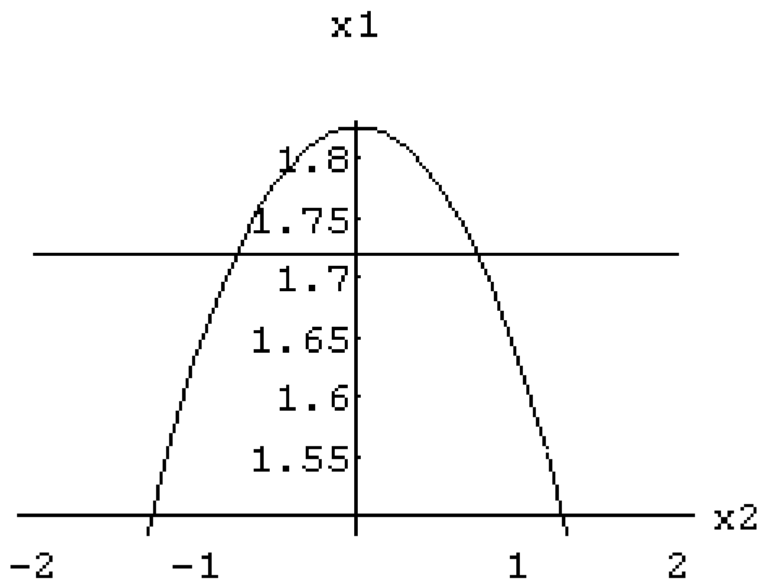
A. To begin our discussion, let us assume that we want to maximize a quadratic utility function subject to an elliptical constraint. Abstracting away from the objective function I want to develop the meaning of the first order Taylor expansion of this elliptical constraint at a given point. Let the elliptical constraint be

$$g(x) = 10 - 3x_1^2 - 2x_2^2$$

Plotting the implicit function yields



B. Next, we want to linearize this function at a particular point, say $x_2 = .75$. Solving the implicit function at this point yields $x_1 = 1.71998$. This solution can be graphed as



C. The Taylor series expansion of this constraint is then defined as

$$G(x) \approx G(x^*) + \nabla_x G(x^*)\, dx$$

The Jacobian of this expansion defined as

$$\nabla_x G(x) = \begin{pmatrix} \frac{\partial G_1(x)}{x_1} & \frac{\partial G_1(x)}{x_2} & \cdots & \frac{\partial G_1(x)}{x_n} \\ \frac{\partial G_2(x)}{x_1} & \frac{\partial G_2(x)}{x_2} & \cdots & \frac{\partial G_2(x)}{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial G_m(x)}{x_1} & \frac{\partial G_m(x)}{x_2} & \ddots & \frac{\partial G_m(x)}{x_n} \end{pmatrix}$$

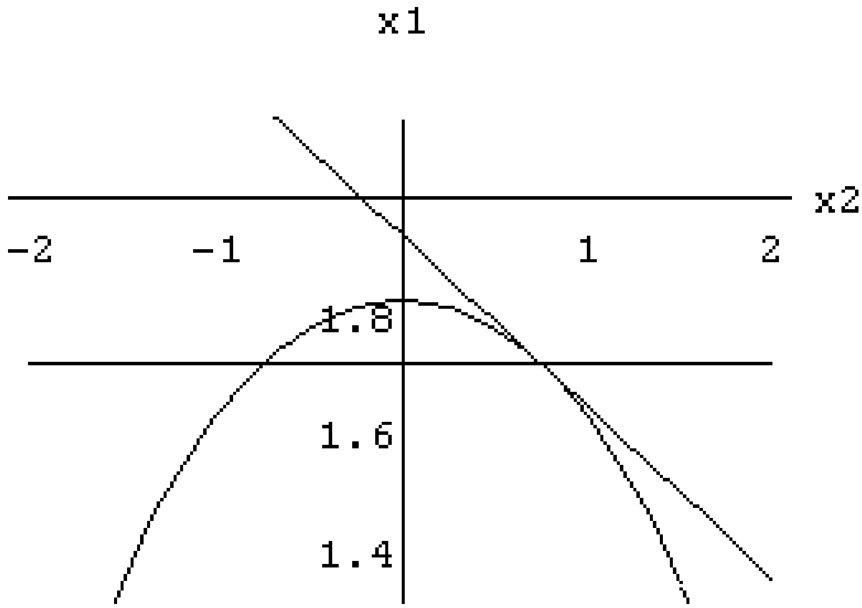
is the cornerstone of the linear approximation. For the current problem, the Jacobian at the point of approximation is

$$\nabla_x G(x) = \begin{pmatrix} -10.3199 & -3.000 \end{pmatrix}$$

D. The linear relationship around that point then becomes

$$\begin{aligned} G(x) - G(x^*) &= \begin{pmatrix} -10.3199 & -3.000 \end{pmatrix} \begin{pmatrix} x_1 - 1.71998 \\ .75 \end{pmatrix} . \\ &= 20 - 10.3199 x_1 - 3.000 x_2 \end{aligned}$$

Graphically, this expansion becomes



II. Optimality Conditions

A. Unconstrained multivariate

$$\left\|\nabla_x f(x)\right\| = 0$$

$$\nabla^2_{xx} f(x) \text{ is positive semidefinite}$$

B. Linearly constrained multivariate

$$Ax = b$$

$$\left\|Z'\nabla_x f(x)\right\| = 0 \text{ or } \nabla_x f(x) = A'\lambda$$

$$Z'\nabla^2_{xx} f(x) Z \text{ is positive semidefinite}$$

C. Linearly constrained inequalities

$$Ax \geq b \text{ with } \hat{A}x = \hat{b}$$

$$\left\|Z'\nabla_x f(x)\right\| = 0 \text{ or } \nabla_x f(x) = \hat{A}'\lambda$$

$$\lambda \geq 0 \text{ for the active constraints}$$

$$Z'\nabla^2_{xx} f(x) Z \text{ is positive semidefinite}$$

D. Nonlinearly constrained inequalities

$$G(x) = 0$$

$$Z(x)'\nabla_x f(x) = 0 \text{ or } \nabla_x f(x) = A(x)'\lambda$$

$$Z(x)'W(x,\lambda)Z(x) \text{ is positive semidefinite}$$

$$W(x,\lambda) = \nabla^2_{xx} f(x) - \sum_{t=1}^{m} \lambda_i \nabla^2_{xx} G_i(x)$$