Lecture VIII
Linear Inequality and Nonlinear Equality Constraints

I. A Review of Linear Equality Constraints

A. As previously developed, the first order necessary conditions for a linearly constrained equality can be written as

$$Z'\nabla_x f(x) = 0$$

An alternative specification for this constraint is

$$\nabla_x f(x) = A'\lambda$$

where λ are a vector of lagrange multipliers. Another way to look at the problem is that the lagrange multipliers are that set of constants which equate the gradient with the linear constraints.

B. To offer an intuitive proof

$$L = f(x) + \lambda_1 (b_1 - a_1 x) + \lambda_2 (b \ - a_2 x)$$
$$\nabla_x L = \nabla_x f(x) - \lambda_1 a_1 - \lambda_2 a_2 = 0 \quad .$$

Thus,

$$\begin{aligned} \nabla_x f(x) &= \lambda_1 a_1 + \lambda_2 a_2 \\ &= A'\lambda \end{aligned} \quad .$$

The general concept of the lagrange multiplier and the null space matrix then work together.

C. Another concept that I want to discuss is the fact that a equality constrained problem can have lagrange multipliers of either side. Intuitively, the lagrange multiplier is the change in the objective function associated with a one unit increase in the right hand side of the constraint. Therefore, in a maximization problem:

1. If the constraint cuts the frontier below the global maximum (assuming that the objective function resembles a quadratic function), the lagrange multiplier will take on a positive value implying that an increase in the right hand side of the constraint will increase the objective function value.
2. Similarly, if the constraint cuts the frontier above the global maximum (again assuming that the objective function resembles a quadratic function), the lagrange multiplier will take on a negative value implying that an increase in the right hand side of the constraint will cause the objective function to decline.

D. A numerical example based on the gradients from last lecture.

1. In the preceding lecture we developed a portfolio model solved for the minimum variance of a portfolio restricting income to 7 percent and using a portfolio balance equation. The optimal portfolio was

$$x = \begin{pmatrix} .11613 \\ -.10666 \\ .39508 \\ .59546 \end{pmatrix}$$

The constraint matrix for this problem was

$$A=\begin{pmatrix} 8.199 & 11.366 & 6.298 & 8.014 \\ 1.0 & 1.0 & 1.0 & 1.0 \end{pmatrix}.$$

Forming the lagrange expression, we will try to find λ_1 and λ_2 such that

$$\nabla_x f(x^*)=\begin{pmatrix} 437.712 \\ 388.069 \\ 467.511 \\ 440.613 \end{pmatrix}=\begin{pmatrix} 8.199 & 1.0 \\ 11.366 & 1.0 \\ 6.298 & 1.0 \\ 8.014 & 1.0 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

2. The two shadow values can be found by solving the system

$$\begin{pmatrix} 8.199 & 1.0 & 436.712 \\ 11.366 & 1.0 & 388.069 \\ 6.298 & 1.0 & 467.511 \\ 8.014 & 1.0 & 440.613 \end{pmatrix}$$

which yields a row-reduced form of

$$\begin{pmatrix} 1 & 0 & -15.675 \\ 0 & 1 & 566.212 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

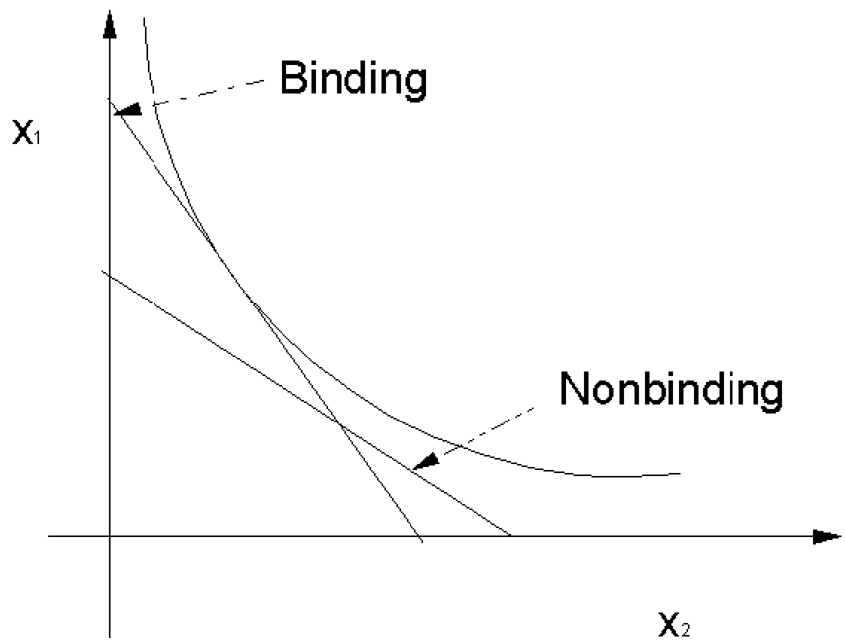
One cautionary note is that using automatic row-reduction procedures in Gauss or Mathematica may reduce the problem beyond the first two row operations due to rounding problems. I would recommend manual reductions.

II. The Case of Inequality Constraints

A. The conditions for inequality constraints are built from the concepts in the linear equality constraints. Specifically, the general form of the LIP problem is stated

$$\begin{aligned} &\max_x f(x) \\ &st\ Ax \leq b \end{aligned}.$$

Again, we will use the null space, but in applying the null space we will divide the set of linear constraints into active or binding and nonactive or nonbinding constraints. Graphically,



The binding constraints must satisfy all the properties that an equality must satisfy, but a nonbinding constraint is omitted from the construction of the null space.

B. Mathematically, the necessary conditions for a linear inequality constrained problem

$$\max_x f(x)$$
$$st\ Ax \le b$$

are:

1. $Ax \le b$
2. $Z'\nabla_x f(x) = 0$ or $\nabla_x f(x) = A'\lambda$
3. $\lambda \ge 0$
4. $Z'\nabla^2_{xx} f(x) Z$ is negative semidefinite

where Z (the null space matrix is defined for only the active or binding constraints of A).

C. Given these conditions we see that the portfolio problem discussed above is not consistent with income greater than or equal to 7 percent rather than exactly equal 7 percent. Specifically, λ_1 = -15.675. Thus, the income constraint is constraining income below the global minimum for variance.

D. Another example developed in the mathematica notebook Constrained2.ma is the utility maximization problem where the consumer maximizes utility defined as:

$$\max_x \begin{pmatrix} 4 & 3 & 1 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} - \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}\begin{pmatrix} 1 & .025 & -.0125 & .0166 \\ .025 & .75 & .0005 & -.001 \\ -.0125 & .0005 & .25 & .045 \\ .0166 & -.001 & .045 & 1.25 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Subject to the income constraint

$$\begin{pmatrix} 3 & 2 & 4 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \le Y .$$

1. We see that the unconstrained problem has a global maximum in the positive quadrant. The exact maximum is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1.91157 \\ 1.94001 \\ 1.45741 \\ 3.52385 \end{pmatrix} .$$

The required level of income to meet this global maximum is 33.0636.

2. Thus, we want to evaluate two scenarios. Under the first scenario the level of income is set at 40.0 which is beyond the level required for the global maximum.
   a. Under this scenario we solve

$$\max_x f(x)$$
$$st \begin{pmatrix} 3 & 2 & 4 & 5 \end{pmatrix} x = 40$$

   which yields an optimal solution of

$$x = \begin{pmatrix} 2.13964 \\ 2.12981 \\ 2.61004 \\ 3.77626 \end{pmatrix}$$

   and an optimal λ of -.14383.
   b. Following the classical necessary conditions for optimization, we see that this point fails even though the projected gradient is zero because the sign on the Lagrange multiplier is wrong.
   c. The utility maximizing point in this case is actually the global maximum.
3. The second scenario then involves constraining y to be less than 30.
   a. In this case, the optimum solution becomes

$$x = \begin{pmatrix} 1.81084 \\ 1.85618 \\ 0.94317 \\ 3.41237 \end{pmatrix}$$

with κ of 0.065537.

b. Under this scenario, it is possible to show that (1) the constraint set is met, (2) the projected gradients vanish, (3) the Lagrange multipliers are greater than or equal to zero, and (4) the projected Hessian is negative definite. Thus, this solution meets the first and second order necessary and sufficient conditions for a maximum.