Lecture VII
Linear Constraints

I. Linearly Constrained Equality

A. The general programming problem discussed in the previous lecture:

$$\max_x \; f(x)$$
$$st \; G(x) = b$$

can be restructured into the linearly constrained equality problem by

$$\max_x f(x)$$
$$st \; Ax = b \; .$$

B. Mathematically one such problem is the portfolio choice problem where an individual minimizes the variance of an investment portfolio subject to an income constraint:

$$\max_x f(x) = x'\Omega x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= v_{11}x_1^2 + 2v_{12}x_1x_2 + 2v_{13}x_1x_3 + 2v_{14}x_1x_4 + v_{22}x_2^2 + 2v_{23}x_2x_3 + 2v_{24}x_2x_4 + v_{33}x_3^2 + 2v_{34}x_3x_4 + v_{44}x_4^2 \; .$$

$$st \quad m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 = Y^*$$

1. Mathematically, solving this system is based on the null-space of the income constraint. In this case, setting the level of x_2 through x_4 determines the level of x_1:

$$x_1 = Y^*/m_1 - m_2/m_1\, x_2 - m_3/m_1\, x_3 - m_4/m_1\, x_4 \, .$$

We could substitute for this equality into the original objective function

$$f^*(x) = v_{11}\left(Y^*/m_1 - m_2/m_1\, x_2 - m_3/m_1\, x_3 - m_4/m_1\, x_4\right)^2 +$$
$$2v_{12}x_2\left(Y^*/m_1 - m_2/m_1\, x_2 - m_3/m_1\, x_3 - m_4/m_1\, x_4\right) +$$
$$2v_{13}x_3\left(Y^*/m_1 - m_2/m_1\, x_2 - m_3/m_1\, x_3 - m_4/m_1\, x_4\right) +$$
$$2v_{14}x_4\left(Y^*/m_1 - m_2/m_1\, x_2 - m_3/m_1\, x_3 - m_4/m_1\, x_4\right) +$$
$$v_{22}x_2^2 + 2v_{23}x_2x_3 + 2v_{24}x_2x_4 + v_{33}x_3^2 + 2v_{34}x_3x_4 + v_{44}x_4^2$$

reducing the multivariate constrained problem to a multivariate unconstrained problem. However, this scenario becomes impractical as the number of equations and variables grow.

C. Another way to conceive the problem is by the formal definition of the null space.
   1. What we want to do is to develop a formalized way of making the statement that $p=(\bar{x}-\hat{x})$ implies that $A\bar{x}=b$ and $A\hat{x}=b$.
   2. Thus, $A\bar{x}=A\hat{x}$ implies that $A(\bar{x}-\hat{x})=0$ or that $Ap=0$. The collection of all such vectors is referred to as the null space.
   3. Specifically, let the columns of matrix Z form a basis; then $AZ=0$ defines the null space

D. In this case, we need to modify our concept of the neighborhood.
   1. Previously, we were interested in any projection around x^*, thus $x=x^*+p$.
   2. Now, we are only interested in those points that maintain feasibility. To accomplish this, we begin with a projection (or perturbation) in the same dimension as the null space. This perturbation is then mapped into the general space using the null space basis. Thus, the set of feasible x's becomes $x=x^*+Zp$.
   3. The second order expansion of the function then becomes

$$f(x)=f(x^*)+\nabla_x f(x^*)Zp+\tfrac{1}{2}p'Z'\nabla^2_{xx}f(x^*)Zp.$$

Again using the mechanics for a maximum, we have that

$$f(x)-f(x^*)=\nabla_x f(x^*)Z\,p+\tfrac{1}{2}p'Z'\nabla^2_{xx}f(x^*)Z\,p\leq 0$$

for any p. Hence, we derive the first necessary condition for a maximum that

$$\nabla_x f(x^*)Z=0$$

which states that the projected gradient is equal to zero at optimal. Similarly, the second order necessary condition for an optimal becomes

$$Z'\nabla^2_{xx}f(x^*)Z \text{ is negative definite.}$$

E. Unconstrained Optimization–Necessary Conditions
   1. $\left\|\nabla_x f\left(x^*\right)\right\|=0$ or x^* is a stationary point.
   2. $\nabla^2_{xx}f\left(x^*\right)$ is negative semi-definite.

F. Unconstrained Optimization–Sufficient Conditions
   1. $\left\|\nabla_x f\left(x^*\right)\right\|=0$ or x^* is a stationary point.
   2. $\nabla^2_{xx}f\left(x^*\right)$ is negative definite.

G. Linearly Constrained Optimization–Necessary Conditions
   1. $Ax^*=b$ or the point is feasible.

2. $\left\| Z'\nabla_x f\left(x^*\right)\right\| = 0$ or the projected gradient is zero. Also equivalently $\nabla_x f\left(x^*\right) = A\lambda^*$ where λ^* is the vector of shadow values.
3. $Z'\nabla^2_{xx} f\left(x^*\right) Z$ is negative semi-definite.

H. Linearly Constrained Optimization–Sufficient Conditions
1. $Ax^* = b$ or the point is feasible.
2. $\left\| Z'\nabla_x f\left(x^*\right)\right\| = 0$ or the projected gradient is zero. Also equivalently $\nabla_x f\left(x^*\right) = A\lambda^*$ where λ^* is the vector of shadow values.
3. $Z'\nabla^2_{xx} f\left(x^*\right) Z$ is negative definite.

II. An Applied Example

A. Using data from Levy and Sarnat, the variance matrix in the above problem can be set as

$$\Omega = \begin{pmatrix} 924.41 & 458.52 & 202.22 & 135.22 \\ 458.52 & 761.29 & 452.99 & 72.55 \\ 202.22 & 452.99 & 490.11 & 109.09 \\ 135.22 & 72.25 & 109.09 & 284.17 \end{pmatrix}.$$

1. Next, imposing two constraints, the first for income and the second for portfolio balance:

$$\begin{aligned} 8.119\, x_1 + 11.366\, x_2 + 6.298\, x_3 + 8.014\, x_4 &= 7.000 \\ x_1 + \quad x_2 + \quad x_3 + \quad x_4 &= 1.000 \end{aligned}.$$

2. Using Quadratic1.f and Quadratic1.SPECS available on the network, the optimum portfolio of

$$x = \begin{pmatrix} .11613 \\ -.10666 \\ .39508 \\ .59546 \end{pmatrix}.$$

a. Summing the vector of x's we see that the sum is 1.0 and the expected return is 7.0. The necessary conditions for optimality then become the projected gradient and hessian matrices. The null space matrix for the constraint set is

$$Z = \begin{pmatrix} -.86287 & 0.0000 \\ .22195 & -.271782 \\ .340534 & -.530893 \\ .300383 & .802675 \end{pmatrix}.$$

b. The gradient vector evaluated a the solution is

$$\nabla_x f(x) = \begin{pmatrix} 437.714 \\ 388.076 \\ 467.519 \\ 440.616 \end{pmatrix}.$$

c. The projected gradient then becomes

$$Z' \nabla_x f(x) = \begin{pmatrix} .0036136 & -.00288491 \end{pmatrix}$$