Lecture VI
An Introduction into Multivariate Unconstrained and Constrained Optimality

I. The development of the first and second order necessary conditions for the multivariate function.

Like the univariate case, the development of the mulutivariate optimum is based on the Taylor series expansion. The multivariate form of the Taylor series expansion is typically restricted to the first two moments by convience, as

$$f(x) = f(x^*) + \nabla_x f(x^*) dx + \tfrac{1}{2} dx' \nabla^2_{xx} f(x^*) dx .$$

Again subracting f(x^*) from each side we are left with

$$f(x) - f(x^*) = \nabla_x f(x) dx + \tfrac{1}{2} dx' \nabla^2_{xx} f(x^*) dx \leq 0$$

for a maximum at x^*.

II. Development of the Constrained Optimum

The first stage in our discussion will be a fairly general description of the problem at hand. We are now interested in developing the optimality conditions for the problem

$$\begin{aligned} &\max_x \; f(x) \\ &st \quad G(x) = b \end{aligned}$$

where f(x) is a scalar valued function mapping

$$G(x) \approx G(x^*) + \nabla_x G(x^*) \, dx$$

$$\nabla_x G(x) = \begin{pmatrix} \frac{\partial G_1(x)}{\partial x_1} & \frac{\partial G_1(x)}{\partial x_2} & \cdots & \frac{\partial G_1(x)}{\partial x_n} \\ \frac{\partial G_2(x)}{\partial x_1} & \frac{\partial G_2(x)}{\partial x_2} & \cdots & \frac{\partial G_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial G_m(x)}{\partial x_1} & \frac{\partial G_m(x)}{\partial x_2} & \ddots & \frac{\partial G_m(x)}{\partial x_n} \end{pmatrix} .$$

Given the Taylor series expansion of the vector equation, we can see that starting from a feasible point $\bar{x}$ and stepping to another feasible point $\hat{x}$ involves solving the equation

$$G(\bar{x}) - G(\hat{x}) = \nabla_x G(\hat{x}) \, dx = 0$$

thus any dx in the null space of $\nabla_x G(\hat{x})$ maintains feasibility.

Simplifying this in the current case assume that $\nabla_x G(\hat{x})\, dx$ can be written as

$$\begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} d\,x_1 \\ d\,x_2 \end{bmatrix} = 0$$

where G_1 and G_2 are the partitioned values from the Jacobian matrix evaluated at $\hat{x}$. This partitioning would be such that G_1 is an m by m matrix where m is the number of equations in the vector valued function. Thus, G_2 is a m by n-m matrix. This partitioning gives us the matrix equation

$$G_1 d\,x_1 + G_2\, d\,x_2 = 0 .$$

Notice that this equation can be solved for $d\,x_1$ as

$$d\,x_1 = -G^{-1}{}_1\, G_2\, d\,x_2$$

which gives the change in x_1 associated with a change in x_2 which maintains feasibility in the constraints.

Returning to the basic optimization problem, we develop our standard Taylor series expansion of the objective function

$$f(x) - f\left(x^*\right) = \nabla_x f(x^*)\, d\,x + \tfrac{1}{2}\, d\,x' \nabla_x^2 f(x^*)\, d\,x \leq 0 .$$

Partitioning this expression in the same manner as the constraint matrix, we have

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} d\,x_1 \\ d\,x_2 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} d\,x_1 & d\,x_2 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} d\,x_1 \\ d\,x_2 \end{bmatrix} \leq 0$$

substituting for the feasibility constraint yields

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} -G_1^{-1}\, G_2\, d\,x_2 \\ d\,x_2 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} -d\,x_2' G_2' G_1^{-1\prime} & d\,x_2' \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} -G_1^{-1}\, G_2\, d\,x_2 \\ d\,x_2 \end{bmatrix} \leq 0 .$$

The first necessary condition for this problem can then be defined as

$$-f_1\, G_1^{-1}\, G_2\, d\,x_2 + f_2\, d\,x_2 = 0$$

$$\left[f_2 - f_1\, G_1^{-1} G_2 \right] d\,x_2 = 0 .$$

$$\left\| f_2 - f_1\, G_1^{-1} G_2 \right\| = 0$$

Intuitively, this condition implies that the gradient projected along a feasible path must be equal to zero at optimum.

The second-order necessary conditions are then defined by

$$\begin{bmatrix} -d\,x_2' G_2' G_1^{-1\prime} & d\,x_2' \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} G_1^{-1} G_2 dx_2 \\ dx_2 \end{bmatrix}$$

$$\begin{bmatrix} dx_2' G_2' G_1^{-1\prime} F_{11} + dx_2' F_{21} & dx_2' G_2' G_1^{-1} F_{12} + dx_2' F_{22} \end{bmatrix} \begin{bmatrix} G_1^{-1} G_2 dx_2 \\ dx_2 \end{bmatrix}$$

$$\left[dx_2' G_2' G_1^{-1\prime} F_{11} G_1^{-1} G_2 dx_2 + dx_2' F_{21} G_1^{-1} G_2 dx_2 + dx_2' G_2' G_1^{-1\prime} F_{12} dx_2 + dx_2' F_{22} dx_2 \right]$$

$$dx_2' \left[G_2' G_1^{-1} F_{11} G_1^{-1} G_2 + F_{21} G_1^{-1} G_2 + G_2' G_1^{-1} F_{12} + F_{22} \right] dx_2$$