

## Lecture IV: The Canonical Form and Null Spaces

- I. To demonstrate the meaning of null space matrices, I first want to discuss the use of canonical forms. A canonical form is a solution to an underidentified system of equations. For example in a crop selection model we may have a land and a labor constraint

$$\begin{aligned}x_1 + x_2 + x_3 + x_4 + x_5 &= 100 \\2x_1 + 3x_2 + x_3 + 4x_4 + x_5 &= 250\end{aligned}$$

- A. The matrix operations used to reduce this matrix to a row reduced form are

$$\begin{aligned}&\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & | & 100 \\ 2 & 3 & 1 & 4 & 1 & | & 250 \end{pmatrix} \\&\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & | & 100 \\ 0 & 1 & -1 & 2 & -1 & | & 50 \end{pmatrix} \\&\begin{pmatrix} 1 & 0 & 2 & -1 & 2 & | & 50 \\ 0 & 1 & -1 & 2 & -1 & | & 50 \end{pmatrix}\end{aligned}$$

1. This representation is a canonical form representing a solution to the system of equations. Specifically,  $x_1=50, x_2=50, x_3=0, x_4=0, x_5=0$  solves the system of equations.
2. In addition, the solution says something about maintaining feasibility in the nonbasic variables. Taking the columns from the nonbasic, or zero-level, variables we can form a homogeneous set of equations ( $Ax=0$ ):

$$\begin{aligned}2x_3 - x_4 + 2x_5 &= 0 \\-x_3 + 2x_4 - x_5 &= 0\end{aligned}$$

This system has the reduced form

$$\begin{aligned}x_3 + x_5 &= 0 \\x_4 &= 0\end{aligned}$$

So any solution where  $x_3=-x_5$  maintains the feasibility of the equations.

- II. A technical definition of the *null space* is the set of solutions of the linear equations  $Ax=0$ . However, a fuller understanding of the null space involves a discussion of dimension

- A. We assume a system of linear equations:

$$\begin{aligned}A &\in M_{n \times m} \\Ax &= b \text{ such that } x \in M_{m \times 1} \\b &\in M_{n \times 1}\end{aligned}$$

by conformatity. In the most interesting null space we have  $n < m$  or that the system is underidentified.

- B. In order to “solve” this system (assuming that the rows of A are linearly independent) we know that we must have at least m nonzero variables in the canonical form.

C. *The null space of the matrix A is the same dimension as the number of free variables.* Taking the first three variables from the above example

$$x_1 + x_2 + x_3 = 100$$

$$2x_1 + 3x_2 + x_3 = 250$$

the reduced form is

$$\left( \begin{array}{ccc|c} 1 & 0 & 2 & 50 \\ 0 & 1 & -1 & 50 \end{array} \right).$$

1. Notice that any change

$$\Delta x_1 + 2\Delta x_3 = 0$$

$$\Delta x_2 - \Delta x_3 = 0$$

is still feasible. For example, if  $\Delta x_3=10 \Rightarrow \Delta x_1=-20$  and  $\Delta x_2=10$

$$x_3 + \Delta x_3 = 10, x_1 + \Delta x_1 = 30, \text{ and } x_2 + \Delta x_2 = 60$$

$$x_1 + x_2 + x_3 = 100$$

$$2x_1 + 3x_2 + x_3 = 250$$

2. The null space is basically a line in 3 space which shows how the variables can be traded to maintain feasibility.

D. Expanding to the first four variables of the original model

$$x_1 + x_2 + x_3 + x_4 = 100$$

$$2x_1 + 3x_2 + x_3 + 4x_4 = 250$$

yields a reduced form

$$\left( \begin{array}{cccc|c} 1 & 0 & 2 & -1 & 50 \\ 0 & 1 & -1 & 2 & 50 \end{array} \right).$$

Again we note that

$$\Delta x_1 + 2\Delta x_3 - \Delta x_4 = 0$$

$$\Delta x_2 - \Delta x_3 + 2\Delta x_4 = 0$$

In this case, the null space is a plane in 4 space. If  $\Delta x_3=1$  and  $\Delta x_4=5$

$$\Delta x_1 = -2\Delta x_3 + \Delta x_4 = 3$$

$$\Delta x_2 = \Delta x_3 - 2\Delta x_4 = -9$$

maintains feasibility.