AEB 6182 Lecture VII
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Farm Portfolio Problem: Part III
Lecture VII

I. Target Motad
   A. The target MOTAD model is a two-attribute risk and return model.
      1. Return is measured as the sum of the expected return of each activity multiplied by the activity level.
      2. Risk is measured as the expected sum of the negative deviations of the solution results from a target-return level.
      3. Risk is then varied parametrically so that a risk-return frontier can be traced out.
   B. Mathematically, the model is stated as

$$\max_x E(z) = \sum_{j=1}^{n} c_j x_j$$

$$st \qquad \sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

$$T - \sum_{j=1}^{n} c_{rj} x_j - y_r \leq 0$$

$$\sum_{r=1}^{n} p_r y_r = l$$

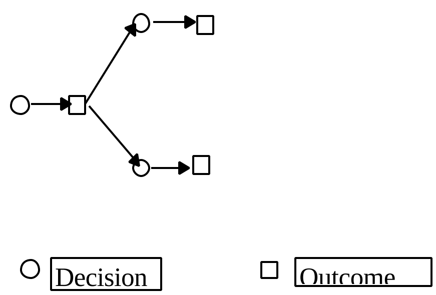
   where:
      1. x_j is the activity level for crop j.
      2. c_j is the expected return on crop j.
      3. a_{ij} is the technical coefficient in column I of row j.
      4. b_j is the right hand side of that technical row.
      5. c_{rj} is the r^{th} outcome for activity j
      6. T is the target loss
      7. y_r is the transfer of the negative deviation
      8. λ is the target loss.
   C. The decision process can then be expressed as a locus of points where the whole farm plan maximizes expected income subject to a target level of negative deviation.

II. Discrete Sequential Stochastic Programming
   A. Target MOTAD, direct expected utility, and even MOTAD begin to develop the concept of constraints being stochastic or met with some level of probability.
      1. In target MOTAD, income under a certain state exceeds the target level of income with some probability.
      2. In direct expected utility maximization the level of wealth transferred to the objective function was represented by a constraint which had some level of probability.
      3. In MOTAD, we minimized the expected negative deviations which implied stochastic constraints.

4. However, in each of these cases, the primary impact of stochastic constraints was on the objective function or some threshold level of risk (as was the case in target MOTAD).

B. The variant of model that we want to develop is referred to as Discrete Sequential Stochastic Programming (DSSP), although other names have been attributed to it. This work grows out of work by Cocks, and focuses on decision processes which are strung out over a discrete number of decision periods.

1. The decision problem can be depicted in a diagram



a. At a discrete point in the future, the farmer has to make a decision, for example a stocking rate on cattle. Given this first round decision and a random outcome, such as rainfall, there is then a subsequent decision to be made, for example whether to sell cattle or buy feed.

b. Each state occurs with a given level of probability and each “node” can contribute to the objective function.

2. A mathematical formulation

$$\max_x \; -c_1x_1 - c_2x_2 - c_3x_3 - c_4x_4 + c_5x_5 - c_6x_6 + c_7x_7 - c_8x_8 - c_9x_9$$

$$-s_{11}x_1 + u_{11}x_2 - f_{11}x_3$$

$$-s_{21}x_1 + u_{11}x_2 \qquad - f_{21}x_4$$

$$- \; x_2 \qquad + \; x_5 - x_6 + x_7$$

$$-s_{12}x_1 \qquad + u_{21}x_7 - f_{12}x_8$$

$$-s_{22}x_1 \qquad + u_{22}x_7 \qquad - f_{22}x_9$$

a. In this model x_1 represents the acres of wheat planted, x_2 is the number of stockers purchased, x_3 the tons purchased under outcome 1, and x_4 the tons of feed under outcome 2.

b. The first two equations, then, simply balance the feed requirements under each state of nature. For example, if there is good rainfall in state 1, then more grazing will be produced by the wheat ,x_2, and less feed will have to be purchased than in state 2. C_1 and c_2 are then the cost of feed in each state weighted by the probability of that state.

c. The third equation then transfers the cattle purchased into the next decision period. X_5 is a variable modeling the number of stockers sold, while x_6 models any additional stockers purchased. The total number of stockers in the next production period is x_7. Given the number of cattle transferred into the next period the feed balance relationships determine the level of feed that must be purchased.

III. Chance Constrained Programming.

A. The DSSP problem above assumes that the possible outcomes can be represented in a finite number of states, although several pieces of applied research have examined the efficiency of approximating the moments of a continuous distribution with a finite number of points.

B. An alternative would be to constrain the probability. For example, assume that you want to constrain the probability that profit will be less than a fixed level T (to borrow the target MOTAD concept). Mathematically, this constraint becomes:

$$P\left[X < x^*\right] \leq T$$

1. Under normality, we can transform this constraint via the confidence interval:

$$x'm - a_{.05}\sqrt{x'\Omega x} \geq T$$

$$a_{.05} \geq \frac{x'm}{\sqrt{x'\Omega x}} - \frac{T}{\sqrt{x'\Omega x}}$$

2. An alternative way to write this constraint is

$$\sum_j x_{ij} m_{ij} + D^* \left[\left(\sum_j s_{ij}^2 x_j^2\right)^{1/2}\right] \leq b_i$$

IV. A Reformulation of the EV Problem

A. The typical mean-variance crop selection model is expressed as

$$\begin{aligned} \max_x \quad & E(p)'x - \frac{r}{2}x'\Sigma x \\ st \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

As we have discussed previously, this specification adjusts for risk in the objective function, but only requires the constraints to be met at the mean.

B. An extension of this model involves appending a term on the constraints which accounts for risk in the constraints. Specifically, rephrasing the profit function as

$$p = p'x - s'v = d'z$$

$$d = \begin{pmatrix} p \\ s \end{pmatrix}$$

$$z = \begin{pmatrix} x \\ y \end{pmatrix}$$

Expected profit is then distributed

$$p \sim N(E(p), z'\Sigma z) \sim N(E(p)'x - E(s)y', x'\Sigma_p x + y'\Sigma_s y)$$

$$\Sigma = \begin{pmatrix} \Sigma_p & 0 \\ 0 & \Sigma_s \end{pmatrix}$$

In this case, the vector s represents the shadow values of the resource constraint. Thus, s'v is the "cost " using the resources.

1. This specification gives rise to a related pair of mathematical programming models.
   a. The primal

$$\max_x \; E(p)'x - \frac{r}{2}x'\Sigma_p x - \frac{r}{2}y'\Sigma_s y$$
$$Ax - r\Sigma_s \le E(s)$$
$$y \ge 0, x \ge 0$$

   b. The dual

$$\min_x E(s)'y + \frac{r}{2}y'\Sigma_s y + \frac{r}{2}x'\Sigma_p x$$
$$A'y + r\,\Sigma_p x \ge E(p)$$
$$y \ge 0, x \ge 0$$

2. This specification is consistent with chance constrained programming. Specifically, maximizing the primal above can be viewed as maximizing the certainty equivalent of a risky revenue subject to the constraint that the probability of the marginal value of the constraints is less than a given critical level with some probability. Mathematically,

$$\Pr\left[y'Ax - y's \le 0\right] \ge \alpha$$

$$\Pr\left[\left(-s'x + E(s)'y\right)/\left(y'\Sigma_s y\right)^{1/2} \le t_s\right] = \alpha$$

$$\Pr\left[E(s)'y - t_s\left(y'\Sigma_s y\right)^{1/2} \le s'y\right] = \alpha$$

   a. Portions of this derivation are dependent on the Kuhn-Tucker conditions. Specifically given the augmented objective function:

$$L = E(p)'x - \frac{r}{2}x'\Sigma_p x - \frac{r}{2}y'\Sigma_s y + y'\left[E(s) + r\,\Sigma_s y - Ax\right]$$

The Kukn-Tucker conditions for a maximum become

$$\frac{\partial L}{\partial x} = E(p) - r\,\Sigma_p x - A'y \leq 0$$

$$x'\frac{\partial L}{\partial x} = x'E(p) - rx'\Sigma_p x - x'A'y = 0$$

$$\frac{\partial L}{\partial y} = E(s) + r\,\Sigma_s y - Ax \geq 0$$

$$y'\frac{\partial L}{\partial y} = y'E(s) + ry'\Sigma_s y - y'Ax = 0$$

V. Preckel, Paul, V., Allen M. Featherstone, and Timothy G. Baker. "Interpreting Dual Variables with Nonmonetary Objectives." *American Journal of Agricultural Economics* 69(1987): 849-51.

A. We have discussed several variants of mathematical programming which involve direct optimization of expected utility. In this lecture, I want to discuss two specifics about the approach. The first involves interpretation of the dual variables from the optimization process and the second involves the parameterization of the probability space.

B. In our discussion expected value-variance procedures, we covered certainty equivalence formulation of the expected value-variance problem from Featherstone and Moss. This formulation allowed us to interpret the shadow values as changes in the certainty equivalent associated with a one unit change in the right hand side of the resource constraint. However, in the direct utility maximization problem, we have no such direct equivalence with a monetary value.

C. Preckel, Featherstone, and Baker propose a simple chain rule formulation which allows us to derive a monetary value from the relationship between the shadow prices. Specifically, if we take a simple expected utility problem:

$$\max -\frac{1}{3}\exp\left(-\left[x_1^{.5}x_2^{.5} - d\right]\right) - \frac{2}{3}\exp\left(-\left[x_1^{.5}x_2^{.5} - d\right]\right)$$

$$\begin{aligned} st \qquad c\,x_1 - d &= 0 \\ x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

1. In this problem, we maximize the expected utility of profit for a Cobb-Douglas production function. In this scenario, you have a stochastic output price, a fixed level of input 2, and a variable level of input 1. The price of input 1 is c.
2. Each shadow price is a yields the number of additional utils provided by each unit increase in the right-hand side. This solution is not very satisfying.
3. However, noting that the first constraint is denominated in dollars, the inverse of this shadow value gives the dollars per util. Thus, the value of the second constraint can be derived by dividing the shadow value of the second constraint by the shadow value of the second constraint.

D. More rigorously,

$$\max \ f(x)$$
$$st \quad g_i(x) = b$$

Writting the optimal level of x as a function of the right hand side and substituting this into the Lagrangian we derive

$$L(x,\pi) = f(x(b)) - \sum_{i=1}^{m} \pi_i(b)\left[g_i(x(b)) - b_i\right]$$

Totally differentiating the objective function with respect to b yields

$$d\,L(b) = \sum_{j=1}^{m}\sum_{k=1}^{n}\left[\frac{\partial\, f\left(x(b)\right)}{\partial\, x_k(b)} - \sum_{i=1}^{m}\pi_i(b)\frac{\partial\, g(x(b))}{\partial\, x_k(b)}\right]\frac{\partial\, x_k(b)}{\partial\, b_j}\,d\,b_j +$$
$$\sum_{j=1}^{m}\sum_{i=1}^{m}\frac{\partial\, \pi_i(b)}{\partial\, b_j}\left[g_i(x(b)) - b_i\right]d\,b_j + \sum_{j=1}^{m}\pi_j(b)\,d\,b_j$$

By the first-order necessary conditions for optimality, this reduces to

$$d\,L(b) = \sum_{j=1}^{m}\pi_j(b)\,d\,b_j$$

Holding the remaining prices constant, we have

$$0 = \pi_i\,d\,b_i + \pi_j\,d\,b_j$$
$$-\frac{d\,b_i}{d\,b_j} = \frac{\pi_j}{\pi_i}$$