Lecture VI: Differentiation

I. Differentials

A. Power Function

1. Generalized power function

$$\frac{d}{dx}cx^n = ncx^{n-1}$$

2. In general

$$\lim_{\Delta x \to 0} \frac{c(x+\Delta x)^n - cx^n}{\Delta x} = \lim_{\Delta x \to 0} c\left[\frac{(x+\Delta x)^n - x^n}{\Delta x}\right]$$

$$= c\lim_{\Delta x \to 0}\left[\frac{(x+\Delta x)^n - x^n}{\Delta x}\right]$$

$$= cnx^{n-1}$$

3. Note for a general function

$$\frac{d}{dx}c\,f(x) = \lim_{\Delta x \to 0}\frac{c\,f(x+\Delta x) - c\,f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} c\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= c\lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= c\,f'(x)$$

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B. Rules of differentiation involving two or more functions of the same variable. Assume two functions $f(x)$ and $g(x)$ are individually differentiable at x.

1. Sum and Difference Rules

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

Reason

$$\frac{d}{dx}\left[f(x)\pm g(x)\right]=\lim_{\Delta x\to 0}\frac{\left[f(x+\Delta x)\pm g(x+\Delta x)\right]-\left[f(x)\pm g(x)\right]}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{\left[f(x+\Delta x)-f(x)\right]\pm\left[g(x+\Delta x)-g(x)\right]}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\pm\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]$$
$$=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]\pm\lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]$$
$$=f'(x)\pm g'(x)$$

a. Examples

(i) $f(x)=x^4\Rightarrow f'(x)=4x^3$

(ii) $f(x)=2x\Rightarrow f'(x)=2$

(iii) $f(x)=2x^2-3x+4\Rightarrow f'(x)=4x-3$

(iv) $f(x)=x^{0.6}\Rightarrow 0.6x^{-0.4}$

(v) $f(x)=\frac{1}{x^3}\Rightarrow f(x)=x^{-3}\Rightarrow f'(x)=-3x^{-4}$

2. Product Rule

$$\frac{d}{dx}\left[f(x)g(x)\right]=f'(x)g(x)+f(x)g'(x)$$

Reason

$$\frac{d}{dx}\left[f(x)g(x)\right]$$
$$=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)+\left[f(x+\Delta x)g(x)-f(x+\Delta x)g(x)\right]}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{\left[f(x+\Delta x)g(x+\Delta x)-f(x+\Delta x)g(x)\right]+\left[f(x+\Delta x)g(x)-f(x)g(x)\right]}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)g(x+\Delta x)-f(x+\Delta x)g(x)}{\Delta x}+\frac{f(x+\Delta x)g(x)-f(x)g(x)}{\Delta x}\right]$$
$$=\lim_{\Delta x\to 0}\left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]+\lim_{\Delta x\to 0}\left[g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$
$$=f(x)g'(x)+f'(x)g(x)$$

By extension

$$\frac{d}{dx}\left[f(x)g(x)h(x)\right]=f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)$$

a. Examples

(i) $$f(x) = x^2(x+4) \Rightarrow f'(x) = 2x(x+4) + x^2 \\ = 2x^2 + 8x + x^2 = 3x^2 + 8x.$$

Note that $f(x) = x^3 + 4x^2 \Rightarrow f'(x) = 3x^2 + 8x$.

(ii) $$f(x) = \frac{(x+4)}{x^2} = (x+4)x^{-2} \\ \Rightarrow f'(x) = x^{-2} + (x+4)(-2x^{-3}) \\ = x^{-2} - 2(x+4)x^{-3} \\ = \frac{x - 2(x+4)}{x^3}$$

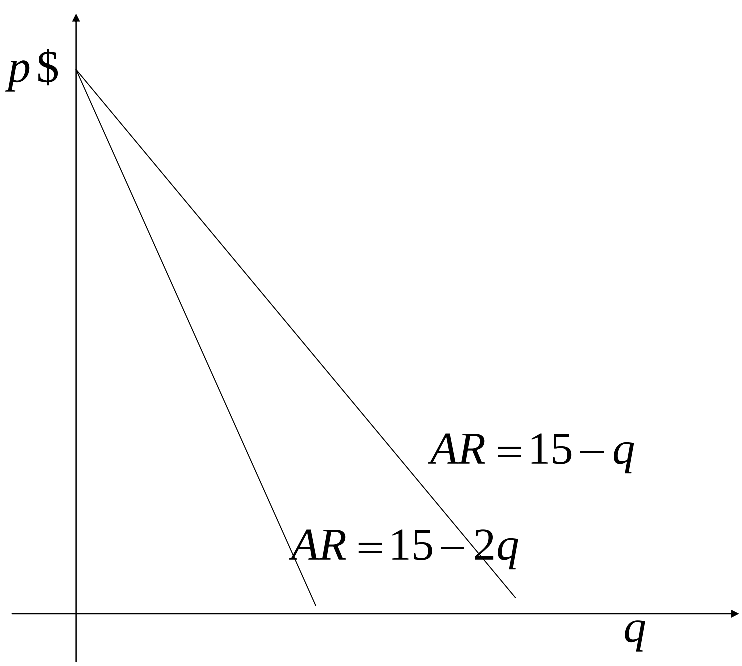
b. Take an average revenue function: AR - Average Revenue, MR - Marginal Revenue, and TR - Total Revenue

$$AR = f(q) = \frac{TR}{q}$$

$$TR = AR \cdot q = f(q) \cdot q$$

$$MR = \frac{d}{dq}TR = \frac{d}{dq}\left[f(q)\cdot q\right] \\ = f'(q)q + f(q)$$

$$MR - AR = f'(q)q + f(q) - f(q) \\ = f'(q)q$$



$$TR = AR \cdot q = (15 - q)q$$
$$= 15q - q^2$$
$$MR = \frac{d}{dq}(15q - q^2)$$
$$= 15 - 2q$$
$$MR = AR + q\frac{d}{dq}AR = (15 - q) + q\frac{d}{dq}(15 - q)$$
$$= 15 - q + q(-1)$$
$$= 15 - 2q.$$

3. Quotient Rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

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a. Examples

(i)
$$\frac{d}{dx}\left(\frac{2x-3}{x+1}\right) = \frac{2(x+1) - (2x-3)(1)}{(x+1)^2}$$
$$= \frac{2x + 2 - 2x + 3}{(x+1)^2} = \frac{5}{(x+1)^2}$$

(ii)
$$\frac{d}{dx}\left(\frac{ax^2 + b}{cx}\right) = \frac{(2ax)(cx) - (ax^2 + b)(c)}{c^2x^2}$$
$$= \frac{2acx^2 - cax^2 - cb}{c^2x^2}$$
$$= \frac{acx^2 - cb}{c^2x^2}$$
$$= \frac{ax^2 - b}{cx^2} = \frac{a}{c} - \frac{b}{cx^2}$$

b. Average and Marginal cost: AC - Average Cost and MC - Marginal Cost.

$$AC = \frac{C(q)}{q}$$
$$\frac{d}{dq}\left(\frac{C(q)}{q}\right) = \frac{C'(q)q - C(q)}{q^2}$$
$$= \frac{1}{q}\left[C'(q) - \frac{C(q)}{q}\right]$$
$$= \frac{1}{q}[MC - AC]$$

Therefore average cost is minimized when the average cost equals the marginal cost:

(i) If $C'(q) > \frac{C(q)}{q} \Rightarrow \frac{d}{dq}\left(\frac{C(q)}{q}\right) > 0$

(ii) If $C'(q) < \frac{C(q)}{q} \Rightarrow \frac{d}{dq}\left(\frac{C(q)}{q}\right) < 0$