Lecture V: Continuity and Differentiability

I. Differentiation and Continuity

A. Review

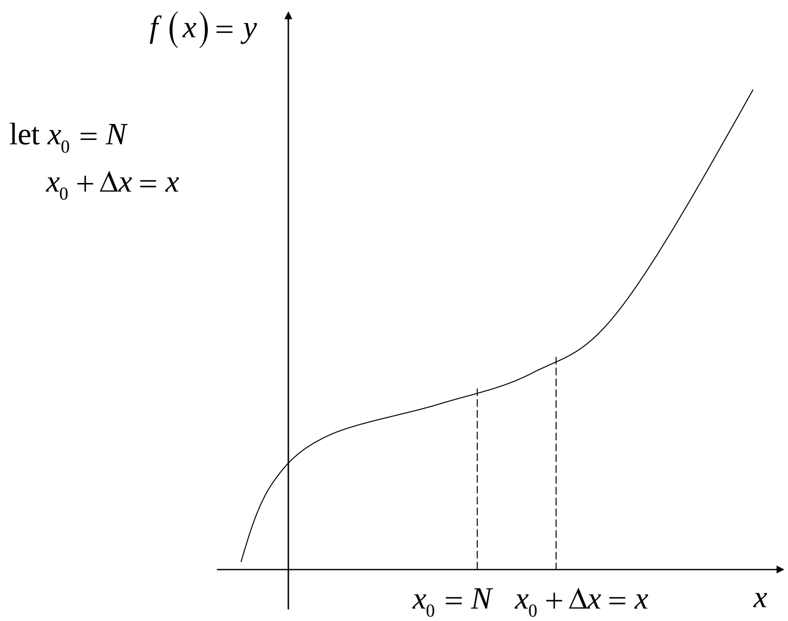
1. Continuity condition

$$\lim_{x \to x_0} f(x) = f(x_0)$$

2. Differentiability condition

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

B. Continuity is a necessary condition for differentiability (but not sufficient)



$x \to N$ becomes analogous to $v \to N$.

1. $\lim_{x \to N} f(x) = f(N)$ {Continuity}.
2. $f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N}$ {Differentiability condition}. We want to show that (1) follows from (2).
3. $f(x) - f(N) = \frac{f(x) - f(N)}{x - N}(x - N)$ where $x \neq N$ by definition of the limit. From the left side of this equation

$$\lim_{x \to N}\left[f(x) - f(N) \right] = \lim_{x \to N} f(x) - f(N)$$

From the right side of equation 3

$$\lim_{x\to N}\left[\left(\frac{f(x)-f(N)}{x-N}\right)(x-N)\right]=\lim_{x\to N}\frac{f(x)-f(N)}{x-N}\lim_{x\to N}(x-N)$$
$$=f'(N)\left(\lim_{x\to N}x-\lim_{x\to N}N\right)$$
$$=f'(N)(N-N)=0$$

$$\therefore \lim_{x\to N} f(x)-f(N)=0 \text{ if } \lim_{x\to N}\frac{f(x)-f(N)}{x-N}\exists$$

Or the function is continuous at x if it is differentiable at x.

Exercise III.A.1 Exercise 6.7 problem 4.

II. Rules of Differentiation and Comparative Statics

A. Rules of one variable

1. Constant function

$$y=f(x)=b\Rightarrow\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{b-b}{\Delta x}=0$$

Intuition: If a function is a constant then it never changes with a change in the independent variable. Economic application – Fixed cost.

$$C(Q)=C_V(Q)+C_F(Q)$$
$$C_F(Q)=b$$
$$\frac{dC(Q)}{dQ}=\frac{dC_V(Q)}{dQ}+\frac{dC_F(Q)}{dQ}$$
$$\frac{dC(Q)}{dQ}=\frac{dC_V(Q)}{dQ}$$
$$\frac{dC_F(Q)}{dQ}=0.$$

The marginal cost of production doesn't depend on the fixed cost. The optimal production level does not depend on fixed cost in the short-run.

2. Power function

$$y=f(x)=x^n\Rightarrow\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x}$$

First, let's generalize the polynomial

$$(x+y)=x+y$$
$$(x+y)^2=x^2+2xy+y^2$$
$$(x+y)^3=x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$(x+y)^n=?$$

In order to determine the general case, we introduce the combinatorial

$$\binom{a}{b}=\frac{a!}{b!(a-b)!} \qquad \{a>b\}$$

{The number of combinations of different things taken b at a time}. For example in a lottery draw 6 numbers out of 49

$$\binom{49}{6}=\frac{49\cdot 48\cdot 47\cdot 46\cdot 45\cdot 44\cdot 43!}{(6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)(49-6)!}$$
$$=\frac{49\cdot 48\cdot 47\cdot 46\cdot 45\cdot 44}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}=\frac{1.006835E+10}{720}$$
$$=13,983,820$$

Using the combinatorial formulation

$$(x+y)^n=\sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i$$

Taking $n=4$ as an example

$$(x+y)^4=\sum_{i=0}^{4}\binom{4}{i}x^{4-i}y^i$$
$$\Rightarrow\binom{4}{0}x^4y^0=\frac{4!}{0!(4-0)!}x^4=x^4$$
$$\Rightarrow\binom{4}{1}x^3y^1=\frac{4!}{1!(4-1)!}x^3y=\frac{4\cdot 3!}{(1)\cdot 3!}x^3y=4x^3y$$
$$\Rightarrow\binom{4}{2}x^2y^2=\frac{4!}{2!(4-2!)}x^2y^2=\frac{4\cdot 3\cdot 2!}{(2\cdot 1)\cdot 2!}=\frac{12}{2}x^2y^2=6x^2y^2$$
$$\Rightarrow\binom{4}{3}x^1y^3=\frac{4!}{3!(4-3)!}xy^3=\frac{4\cdot 3!}{3!\cdot(1)}=4xy^3$$
$$\Rightarrow\binom{4}{4}x^0y^4=\frac{4!}{4!(4-4)!}y^4=\frac{4!}{4!\cdot(1)}xy^4=y^4$$
$$(x+y)^4=\sum_{i=0}^{4}\binom{4}{i}x^{4-i}y^i=x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

Note:

$$\sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^{i}=\binom{n}{0}x^{n}y^{0}+\binom{n}{1}x^{n-1}y^{1}+\sum_{i=2}^{n}\binom{n}{i}x^{n-i}y^{i}$$

$$\binom{n}{0}=\frac{n!}{n!(n-n)!}=\frac{n!}{n!\cdot(1)}=1$$

$$\binom{n}{1}=\frac{n!}{(n-1)!(n-[n-1])!}=\frac{n\cdot(n-1)!}{(n-1)!\cdot(1)}=n$$

$$\therefore\sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^{i}=x^{n}+nx^{n-1}y+\sum_{i=2}^{n}\binom{n}{i}x^{n-i}y^{i}$$

$$=x^{n}+nx^{n-1}y+y^{2}\sum_{i=2}^{n}\binom{n}{i}x^{n-i}y^{i-2}$$

Let $y=\Delta x\Rightarrow f(x)=x^{n}$, $f(x+\Delta x)=(x+\Delta x)^{n}$

$$f(x+\Delta x)=x^{n}+nx^{n-1}(\Delta x)+(\Delta x)^{2}\sum_{i=2}^{n}\binom{n}{i}x^{n-i}(\Delta x)^{i-2}$$

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^{n}+nx^{n-1}(\Delta x)+(\Delta x)^{2}\sum_{i=2}^{n}\binom{n}{i}x^{n-i}(\Delta x)^{i-2}-x^{n}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}nx^{n-1}+(\Delta x)\sum_{i=2}^{n}\binom{n}{i}x^{n-i}(\Delta x)^{i-2}-x^{n}$$

$$=nx^{n-1}$$

3. Generalized power functions

$$\frac{d}{dx}cx^{n}=ncx^{n-1}$$

Exercise III.A.2 7.1 problem 2 and 3 p. 159.