

## Lecture IV: Limits and Differentiation

### I. Review of Inequalities and Absolute Values

#### A. Rules of inequalities

1. Inequalities are transitive

$$a > b, b > c \Rightarrow a > c$$

$$a \geq b, b \geq c \Rightarrow a \geq c$$

Continued inequality

$$a > b > c$$

$$a > b > c > d$$

2. Rules

a. Rule I:  $a > b \Rightarrow a \pm k > b \pm k$

b. Rule II:  $a > b \Rightarrow \begin{cases} ka > kb & (k > 0) \\ ka < kb & (k < 0) \end{cases}$

c. Rule III:  $a > b \Rightarrow a^2 > b^2 \quad (b \geq 0)$

#### B. Absolute Values and Inequalities

$$|x| < n \Leftrightarrow -n < x < n \quad (n > 0)$$

1. Properties

a.  $|m| + |n| \geq |m + n|$ ,  $m$  and  $n$  may have different signs

b.  $|m||n| = |mn|$

#### C. Solution of an inequality

$$3x - 3 > x + 1$$

1.  $2x > 4$

$$x > 2$$

$$|1 - x| \leq 3$$

2.  $-3 \leq 1 - x \leq 3$

$$-4 \leq -x \leq 2$$

$$4 \geq x \geq -2$$

II.C.2 Exercise 6.5 problem 1 and 2 p. 144.

### II. Limit Theorems

A. Theorem I: If  $q = av + b$  then

$$\lim_{v \rightarrow N} q = aN + b$$

where  $a$  and  $b$  are constants.

B. Theorem II: If  $q = g(v) = b$  then

$$\lim_{v \rightarrow N} q = b$$

C. Theorem III: If  $q = v$  then  $\lim_{v \rightarrow N} q = N$ . If  $q = v^k$  then  $\lim_{v \rightarrow N} q = N^k$ .

D. Theorems involving two functions with

$$\lim_{v \rightarrow N} q_1 = L_1 \text{ and } \lim_{v \rightarrow N} q_2 = L_2$$

with both  $L_1$  and  $L_2$  finite.

1. Theorem IV:  $\lim_{v \rightarrow N} (q_1 \pm q_2) = L_1 \pm L_2$ .

2. Theorem V:  $\lim_{v \rightarrow N} (q_1 q_2) = L_1 L_2$

II.C.3. Exercise 6.6 Chain problem 1,2,3.

### III. Continuity and Differentiability of a function

A. Continuity of a function.

1. For a function to be continuous it must have three properties at  $N$ .

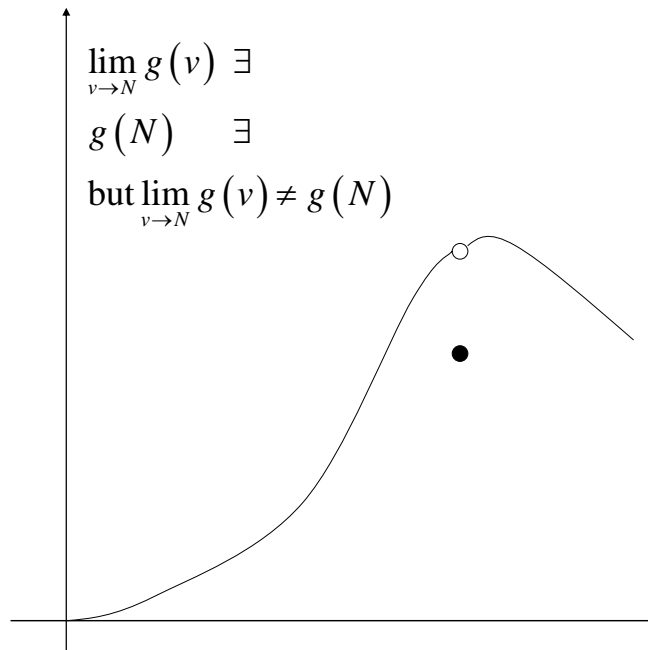
a. The point  $N$  must be in the domain of the function ( $g(N)$  is defined).

b. The function must have a limit as  $v \rightarrow N$ :

$$\lim_{v \rightarrow N} g(v) \exists.$$

c. The limit must equal the value of the function

$$\lim_{v \rightarrow N} g(v) = g(N)$$



B. Polynomial and Rational Functions

1. Polynomial functions are continuous at every point in their domain.

2. Rational functions are continuous at every point that is in the functions domain.

$$q = \frac{v^3 + v^2 - 4v - 4}{v^2 - 4}$$

Is continuous except where  $v = \pm 2$  which are not in the function's domain.

C. Differentiability of a function

1. Continuity condition

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

2. Differentiability condition

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \end{aligned}$$

3. Continuity is a necessary condition for differentiability (but not sufficient).

$$\left. \begin{array}{l} x_0 = N \\ x_0 + \Delta x = x \end{array} \right\} \Rightarrow \begin{array}{l} \Delta x = x_0 + \Delta x - x_0 \\ \Delta x = x - N \end{array}$$

$x \rightarrow N$  becomes analogous to  $x \rightarrow N$ .

a.  $\lim_{x \rightarrow N} f(x) = f(N)$ .

b.  $f'(N) = \lim_{x \rightarrow N} \frac{f(x) - f(N)}{x - N}$ .

c. We want to show that a follows from b.

d.  $f(x) - f(N) = \frac{f(x) - f(N)}{x - N} (x - N)$  with  $x \neq N$  by the definition of the limit.

$$\lim_{x \rightarrow N} [f(x) - f(N)] = \lim_{x \rightarrow N} f(x) - f(N) \text{ since } f(N) \text{ is a constant}$$

$$\begin{aligned} \lim_{x \rightarrow N} \left[ \left( \frac{f(x) - f(N)}{x - N} \right) (x - N) \right] &= \lim_{x \rightarrow N} \frac{f(x) - f(N)}{x - N} \lim_{x \rightarrow N} (x - N) \\ &= f'(N) \left( \lim_{x \rightarrow N} x - N \right) \\ &= f'(N) (N - N) = 0 \end{aligned}$$

$$\therefore \lim_{x \rightarrow N} f(x) - f(N) = 0 \text{ if } \lim_{x \rightarrow N} \frac{f(x) - f(N)}{x - N} \exists.$$

4. Although differentiability implies continuity, the reverse is not true

$$q = f(x) = |x - 2| + 1$$