

Lecture III: Derivatives

I. Functions (Continued).

A. Types of functions:

1. Constant function
2. Polynomial function
3. Rational Functions

$$y = \frac{x-1}{x^2 + 2x + 4}$$

4. Nonalgebraic functions

$$y = b^x$$

$$y = \log_b x$$

Nonalgebraic functions are also called transcendental functions.

B. Homogenous functions: If each coefficient in a multivariate function is multiplied by the same, non-negative, constant then a homogenous function will return a value equal to the original value times the same constant raised to some power.

$$f(\lambda x, \lambda y) = \lambda^k f(x, y)$$

1. This function is homogenous of degree k .

$$f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$$

$$\begin{aligned}\Rightarrow f(\lambda x_1, \lambda x_2) &= (\lambda x_1)^\alpha (\lambda x_2)^{1-\alpha} \\ &= \lambda^\alpha \lambda^{1-\alpha} x_1^\alpha x_2^{1-\alpha} \\ &= \lambda x_1^\alpha x_2^{1-\alpha}\end{aligned}$$

$$f(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$$

$$\begin{aligned}\Rightarrow f(\lambda x_1, \lambda x_2) &= (\lambda x_1)^2 + 2(\lambda x_1)(\lambda x_2) + (\lambda x_2)^2 \\ &= \lambda^2 x_1^2 + 2\lambda^2 x_1x_2 + \lambda^2 x_2^2 \\ &= \lambda^2 (x_1^2 + 2x_1x_2 + x_2^2)\end{aligned}$$

II. Derivatives

A. Theory

1. How many times in economics are we interested in changes?

a. Microeconomics

(I) Marginal cost

(II) Marginal return

(III) Marginal utility

(IV) Marginal rate of substitution

(V) Marginal product

(VI) Marginal rate of technical substitution

b. Macroeconomics

(I) Marginal propensity to consumer

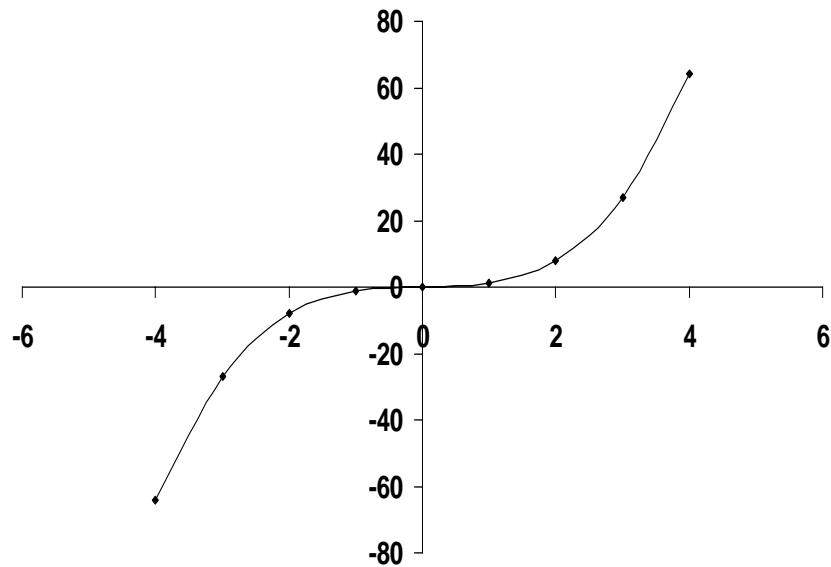
2. Difference quotient

$$y = f(x)$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Think of this as the slope of the function or rise over run. Let

$$f(x) = x^3$$



$$x = 0, \Delta x = 2$$

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

$$= \frac{(x^2 + 2x(\Delta x) + (\Delta x)^2)(x + \Delta x) - x^3}{\Delta x}$$

$$= \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$

$$= \frac{3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} = 3x^2 + 3x(\Delta x) + (\Delta x)^2$$

$$x = 0, \Delta x = 2 \Rightarrow \frac{\Delta y}{\Delta x} = 3(0)^2 + 3(0)(2) + (2)^2 = 4 \leftarrow \frac{f(2) - f(0)}{2} = \frac{8 - 0}{2}$$

3. The derivative is equal to the difference quotient as Δx approaches 0

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

from above

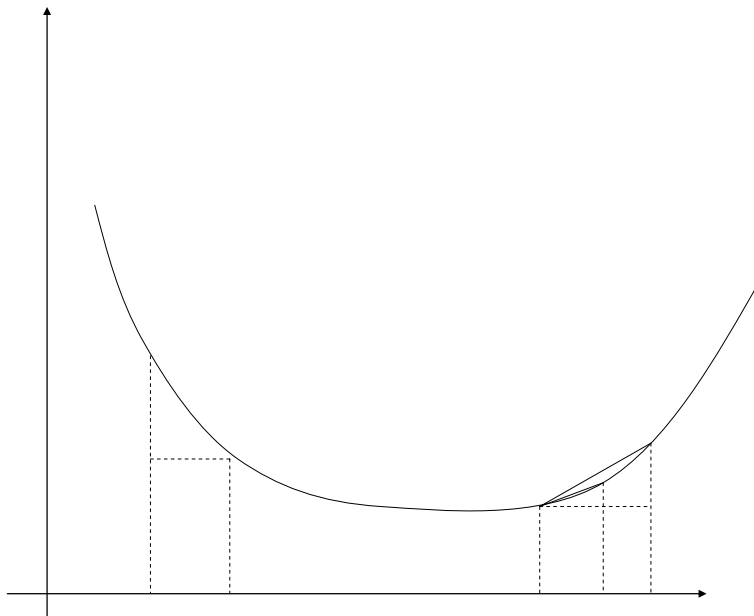
$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} 3x^2 + 3x(\Delta x) + (\Delta x)^2 = 3x^2.$$

- a. Note that the derivative itself is a function. Derivative actually means derived function. In this case $y = f(x)$ is the primary or primitive function.
- b. The derivative is the instantaneous rate of change.
- c. Notation

$$\frac{dy}{dx} = f'(x) = D_x f(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

Exercise IIB, Exercise 6.2 problem 2 Chaing p 131.

B. Slope



1. $\frac{\Delta C(q)}{\Delta q}$ represents an average slope as Δq becomes increasingly smaller,

$$\frac{\Delta C(q)}{\Delta q} \rightarrow \frac{dC(q)}{dq}$$

C. Limits

1. Let

$$q \equiv \frac{\Delta y}{\Delta x}$$

or q be the quotient from the quotient rule, and

$$v \equiv \Delta x$$

or the variation from the quotient rule. Then the limit or derivative is defined as

$$\frac{dy}{dx} = \lim_{v \rightarrow 0} q.$$

2. We define the limit by the question “what values does one variable (say q) approach as another variable (say v) approaches a specific value (say 0)?” In the general case

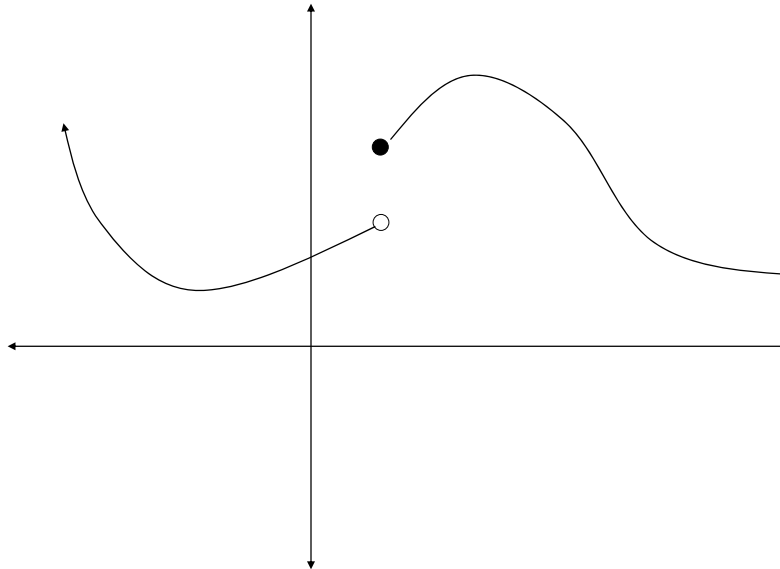
$$q = g(v) \Rightarrow \lim_{v \rightarrow N} q(v).$$

- The left-side limit $v \rightarrow N^-$ from values less than N .
- The right-side limit $v \rightarrow N^+$ from values greater than N .
- When and only when both limits have the same finite value do we consider the limit of q to exist:

$$\lim_{v \rightarrow N} q = 3x^2$$

in the case of the derivative of x^3 .

3. Note that the right and left-side limits need not be the same



4. Examples:

$$\text{a. } q = 2 + v^2 \Rightarrow \left\{ \begin{array}{l} \lim_{v \rightarrow 0} q = 2 \\ \lim_{v \rightarrow 0^-} q \Rightarrow \left\{ \begin{array}{l} 2 + (-1)^2 \\ 2 + (-1/10)^2 \\ 2 + (-1/100)^2 \end{array} \right. \\ \lim_{v \rightarrow 0^+} q \Rightarrow \left\{ \begin{array}{l} 2 + 1^2 \\ 2 + (1/10)^2 \\ 2 + (1/100)^2 \end{array} \right. \end{array} \right.$$

b. $q = \frac{1-v^2}{1-v}$. Note that by the difference of two squares in the numerator. Any time you see the difference of two squares
 $1 - v^2 = (1+v)(1-v)$

or in general

$$(a+b)(a-b) = a^2 + ab - ab - b^2 = a^2 - b^2.$$

Alternatively, we could view this process through synthetic division

$$\begin{array}{r}
 \overline{) 1+0v-v^2} \\
 \underline{-(1-v)} \\
 v-v^2 \\
 \underline{-(v-v^2)} \\
 0
 \end{array}$$

Given this result

$$\lim_{v \rightarrow 1} \frac{1-v^2}{1-v} = \lim_{v \rightarrow 1} 1+v = 2$$

c. $q = \frac{2v+5}{v+1}$. By synthetic division

$$\begin{array}{r} 1+v \\ v+1 \overline{) 2v+5} \\ \underline{-(2v+2)} \\ 3 \end{array} \Rightarrow q = 2 + \frac{3}{v+1}$$

5. Formal view of the limit concept.

a. Neighborhood

(I) For any given number L , there can always be found a number $(L - a_1) < L$ and another number $(L + a_2) > L$ for $a_1, a_2 > 0$.

(II) Closed and open intervals

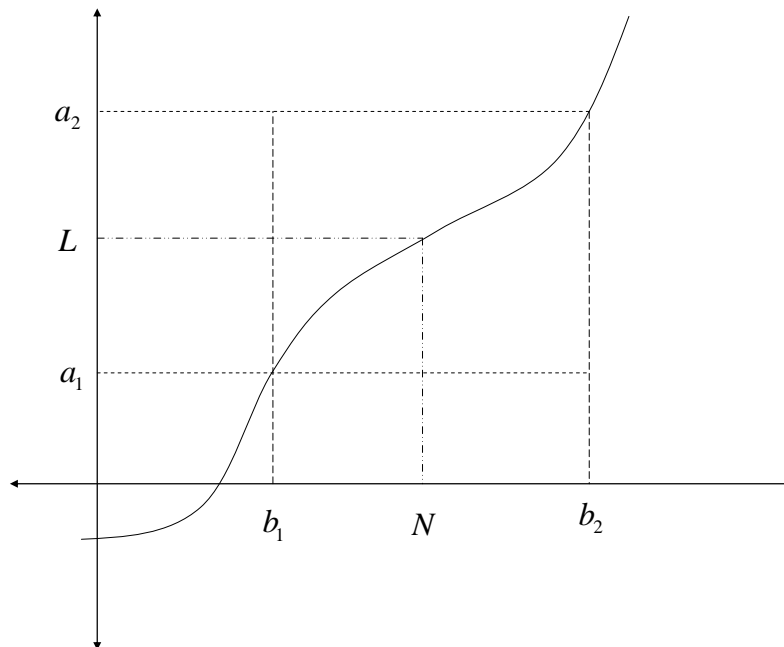
$$[L - a_1, L + a_2] = \{q \mid L - a_1 \leq q \leq L + a_2\}$$

$$(L - a_1, L + a_2) = \{q \mid L - a_1 < q < L + a_2\}$$

(III) A neighborhood of L is the open interval covering the number L

$$(L - a_1, L + a_2).$$

b. As v approaches a number N , the limit of $q = g(v)$ is the number L if for every neighborhood of L that can be chosen however small there can be found a corresponding neighborhood of N (excluding the point $v = N$) in the domain of the function such that for each value of v in the neighborhood, its image lies in the chosen L neighborhood.



Building in the previous example:

$$q = \frac{1-v^2}{1-v} = 1+v \quad \exists: v \neq 1$$

L is the dependent variable $(2-a_1, 2+a_2)$ while N is the independent variable whose value is bounded $(1-b_1, 1-b_2)$.

Thus, $1-b_1 < v < 1+b_2$ must satisfy $2-a_1 < q < 2+a_2$. Since $q = 1+v$

$$2-a_1 < 1+v < 2+a_2$$

$$1-a_1 < v < 1+a_2$$

Thus, any two numbers b_1 and b_2 can be chosen such that $b_1 = a_1$ and $b_2 = a_2$.

II.C.1 Exercise 6.4 problem 2 p. 140