Lecture II: Sets and Functions

I. Ingredients of a Mathematical Model

A. Variables

1. Variable–something whose value can change, take on different values.
2. Solutions values
3. Endogenous variables–variables solved or determined by the model

$$Q_S = 100 - 2p + 4p^2$$
$$Q_D = 1,000 - 10p$$

The market clearing condition is $Q_S = Q_D$ (quantity supplied equals the quantity demand). Solution

$$100 - 2p + 4p^2 = 1,000 - 10p$$
$$-900 + 8p + 4p^2 = 0$$

or

$$4p^2 + 8p - 900 = 0$$

Quadratic formula

$$Ax^2 + Bx + C = 0 \Rightarrow x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Hence,

$$p = \frac{-8 \pm \sqrt{64 - 4(-900)(4)}}{2(4)} = \frac{-8 \pm 120.2664}{8} = 14.0333$$

$$Q_S = 100 - 2(14.0333) + 4(14.0333)^2 = 859.6674$$
$$Q_D = 1,000 - 10(14.0333) = 859.6670$$

Question: which variables are endogenous? Q_S, Q_D, and p are all endogenous.

4. Exogenous variables–those variables whose value is determined outside or external to the mode.
   a. Let y be aggregate income, amend the previous model

$$Q_S = 100 - 2p + 4p^2$$
$$Q_D = 1,000 - 10p + 16y$$

{Question: is this the correct sign for income?}

$$1,000-10p+16y=100-2p+4p^2$$

$$4p^2-2p+10p+100-1,000-16y=0$$

$$4p^2+8p-(900+16y)=0$$

$$A=4$$

$$B=8$$

$$C=-(900+16y)$$

Solving the quadratic formula

$$p=\frac{-8\pm\sqrt{8^2-4(4)(-(900+16y))}}{2(4)}$$

$$=\frac{-8\pm\sqrt{(4\cdot 2)^2-4^2(-(900-16y))}}{2(4)}$$

$$=\frac{-8\pm\sqrt{4^2(4+900+16y)}}{2(4)}$$

$$=\frac{-8\pm 4\sqrt{904+16y}}{8}$$

$$=1+\tfrac{1}{2}\sqrt{4(226+4y)}=1+\sqrt{226+4y}$$

Solving for the quantity

$$Q=1,000-10\left(1+\sqrt{226+4y}\right)+16y$$

$$=900-10\sqrt{226+4y}+16y$$

For a solution to exist

$$226+4y\geq 0$$

$$4y\geq -226$$

$$y\geq -56.5$$

b. In this example y is an exogenous variable.

B. Equations

1. Definitional equation–identifies or defines one variable in terms of another variable or group of variables

$$\pi=R-C\Rightarrow\begin{cases}\pi \text{ profit}\\ R \text{ revenue}\\ C \text{ cost}\end{cases}$$

2. Behavioral equations–specifies the way that variables interact or behave. Such as consumer demand

$$Q_D=1,000-10p+16y$$

3. Equilibrium conditions–requires some notion of equilibrium

$$Q_S=Q_D$$

If the quantity supplied equals the quantity demanded the market is in equilibrium, it is stationary.

C. Concept of Sets
1. Set–A collection of distinct objects.
2. Description of sets
   a. Enumeration $S = \{2,3,4\}$
   b. Description $I = \{x | x \text{ is a positive integer}\}$ or more precisely $I = \{x | x \in \{1,2,3,4,\cdots\}\}$.
3. Relationship between sets
   a. $\subset$ contained in
   b. $\supset$ includes
   c. $\in$ is an element of
   d. $=$ equal to. If $T \subset S$ and $T \supset S$, or if $x \in T$ implies $x \in S$, and $x \in S$ implies $x \in T$, then $T = S$.
4. Set operators
   a. Union $A \cup B = \{x | x \in A \text{ or } x \in B\}$.
   b. Intersection $A \cap B = \{x | x \in A \text{ and } x \in B\}$
   c. Compliment $\tilde{A} = \{x | x \notin A \text{ and } x \in U\}$
5. Laws of operations
   a. Commutative law

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

   b. Distributive law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Exercise I.B.1→Exercise 2.3 a, c, e, g Page 17 Chaing

II. Relationships and Functions

A. Ordered Pairs
1. In set theory $\{a,b\} = \{b,a\}$, but in ordered pairs (as opposed to unordered pairs) $(a,b) \neq (b,a) \ni: a \neq b$. Ordered sets, no reason to stop at ordered pairs. We could have ordered triplets.
2. Cartesian product or direct product

$$x = \{1,2\} \quad y = \{3,4\}$$

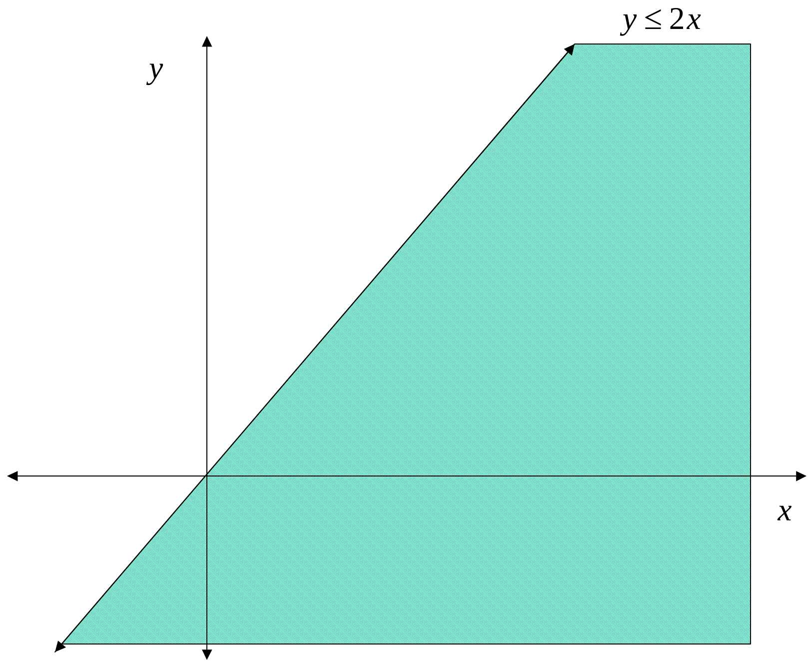
$x * y$ or "x cross y"

$$x * y = \{(1,3),(1,4),(2,3),(2,4)\}$$

$$x * y = \{(a,b) | a \in x \text{ and } b \in y\}$$

B. Relationships and Functions
1. Any collection of ordered pairs–any subset of a Cartesian product–will constitute a relationship between x and y.

2. When x is given in a relationship, y may not be uniquely determined.



3. As a special case, for each value of x there is one and only one value of y.

$$y = f(x)$$

y is a function of x.

   a. A function is also sometimes referred to as a mapping or transformation

$$f : x \rightarrow y$$

   b. $y = f(x)$
      i. x is referred to as the argument of the function, or the independent variable.
      ii. y is referred to as the value of the function or the dependent variable.
   c. The set of all permissible xs is called the domain of the function, while the set of all possible outcomes is called the range. The y value into which a particular x is mapped is called the image of x.

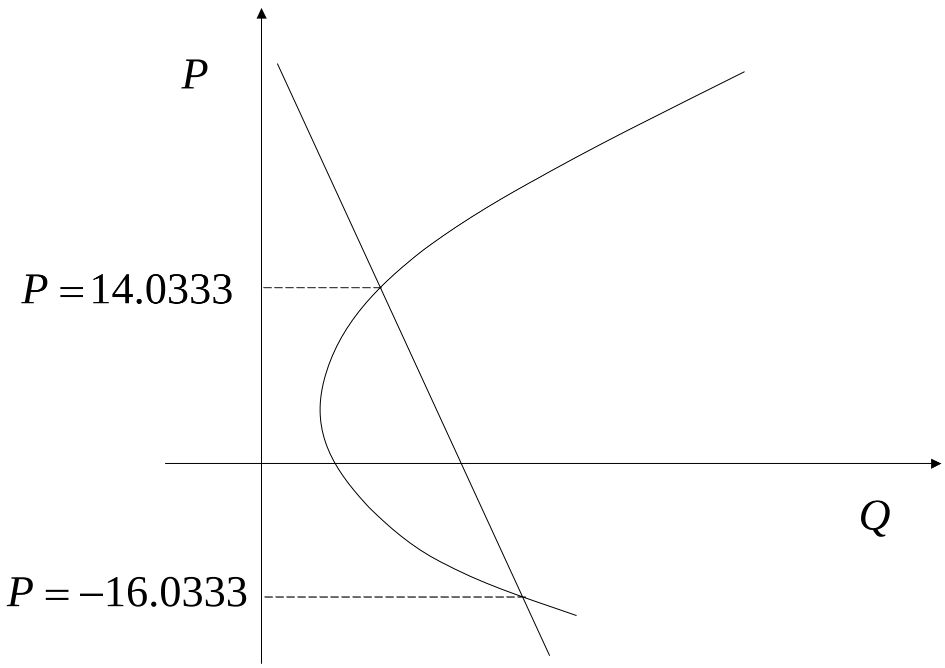
3. Economic applications
   a. Supply and Demand. Earlier we worked with the problem

$$Q_S = 100 - 2P + 4P^2$$
$$Q_D = 1,000 - 10P$$
$$Q_S = Q_D$$

Both equations are behavioral equations relating quantity with price. Solving the two equations yielded

$$P = \frac{-8 \pm 120.2664}{8}$$
$$= -16.0333, 14.0333$$

I implicitly assumed that we wanted a positive price, or I imposed the restriction that the price was greater than or equal to zero. Graphically, the two equations can be depicted as



b. Utility maximization

$$U = x_1^{\alpha} x_2^{\beta} x_3^{1-\alpha-\beta}$$

In general, the utility function is defined over the positive orthant so we are looking for a vector in the positive orthant.